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ANALYTICAL AND EXPERIMENTAL STUDIES OF IMPACT DAMPERS

by

Sami Faiz Masri

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Sami Faiz Masri

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ABSTRACT

A study is made of the general behavior of a single particle impact damper, with the main emphasis on symmetric 2 impacts/cycle motion. The exact solution for this case is derived analytically and its asymptotically stable regions are determined. The stability analysis defines the zones where the modulus of all the eigenvalues of a certain matrix relating conditions after each of two consecutive impacts is less than unity.

Results of the analysis are supplemented and verified by experimental studies with a mechanical model and an analog computer. Additional numerical investigations are made with a digital computer.

It is found that, under practically realizable conditions, impact damping is effective in reducing the vibration amplitude levels resulting from sinusoidal, random, or impulse-like excitation.

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NOMENCLATURE

A = displacement amplitude of primary system in the absence of the impact damper

c = damping constant

d = clearance in which the particle is free to oscillate

e = coefficient of restitution

F_o = maximum force of excitation

k = spring constant

M = mass of primary system

m = mass of particle

P = perturbation matrix

R = remainder term

r = ratio, forcing frequency/natural frequency

t = time

v = absolute velocity of particle

x = displacement of M

x_a = displacement immediately after impact

x_b = displacement immediately before impact

y_1 = displacement of particle

y = relative displacement of particle with respect to M

α = phase angle (initially unknown)

δ = ratio of critical damping

μ = mass ratio, m/M

$\vec{\xi}$ = perturbation vector

ρ = ratio, d/A

ψ = phase angle (due to damping)

τ = phase angle, $\tau = \alpha - \psi$

ω = natural frequency, $\sqrt{k/M}$

Ω = forcing frequency

1. INTRODUCTION

1.1. Development of Impact Damping

The idea of reducing the vibration of a mechanical system by attaching to it a container in which a solid particle is constrained to oscillate was conceived and investigated in 1944 by Lieber and Jensen[(1)*].

In that investigation, the authors assumed that the motion of an undamped single degree of freedom oscillator with an operating impact damper (referred to as an "acceleration damper") was still simple harmonic; that the impact of the primary system with the particle was completely plastic (i. e. no rebound); and that during a period of the sinusoidal forcing function two impacts occur at equal time intervals and at opposite sides of the container (i. e. symmetric 2 impacts/cycle motion). As a result, they determined that for maximum energy dissipation per cycle, the clearance of the particle should be π times the amplitude of response.

Grubin[(2)], in his investigation of this device, assumed the existence of symmetric 2 impacts/cycle motion (henceforth, unless otherwise specified, the motion will be assumed to be symmetric) and he determined the behavior of the viscously damped primary system, by adding the effects of many impacts. Arnold[(3)] investigated experimentally and theoretically a similar system, without viscous damping, by representing the force that acts during impact by a Fourier series.

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*Numbers in parenthesis designate references at end of thesis.

A considerably simpler method for deriving the solution for 2 impacts/cycle motion, which requires only the consideration of two successive impacts, was suggested by Warburton[4], and he used it to obtain the solution for the special case of an undamped primary system forced at resonance.

On the experimental side, the feasibility of using impact damping to reduce the vibrations of such diverse systems as ship hulls, cantilever beams, single degree of freedom systems, and turbine buckets, was investigated by McGoldrick[5], Lieber and Tripp[6], Sankey[7], and Duckwald[8], respectively. Estabrook and Plunkett[9] made an analytical study of impact damping in turbine buckets. Also this type of damping was employed in reducing the vibrations of telephone switching relays.

It is worth mentioning that, according to Ref. 9, Paget made a study of mechanical damping by impact in 1930.

1.2. Remarks on the Analytical Treatment

The assumption common in all the theoretical treatments mentioned above, that of having symmetric 2 impacts/cycle motion, may appear to be an overly restrictive assumption which places unwarranted emphasis on this particular type of motion, and which is made merely to expedite the analytical treatment.

The justification for this assumption, however, is that it is this type of motion that predominates in experimental studies of impact damping, as observed and reported by most investigators in this field.

1.3. Motivation and Scope

Lately, interest in impact dampers appears to have subsided.

This may be related in part to the lack of a complete understanding of the general behavior of this device, and in particular the stability of its periodic motion.

The objective of the present study is to extend and complement the work of other investigators in this field, to assess the practical feasibility of the device, and in particular to determine the stability of 2 impacts/cycle solutions. To this end, the theoretical solution for 2 impacts/cycle motion is derived in Chapter 2, and its stability boundaries are determined in Chapter 3. The experimental studies that were conducted in the course of this investigation with a mechanical model, an electric analog, supplemented by numerical studies with a digital computer, are described and their results interpreted and compared to the theoretical predictions in Chapter 4. Conclusions drawn from this research, and recommendations for future work, are stated in Chapter 5.

2. STEADY STATE SOLUTION

2.1. Existence of Periodic Solutions

Since the motion of any real system (where there is always some damping) is bounded, it can be shown that the governing differential equation possesses periodic solutions (which may or may not be stable). Rather than invoking an existence theorem, the existence of a periodic 2 impacts/cycle solution will be proved by actually constructing one.

2.2. Two-Impacts-Per-Cycle Solution

A model of the system under discussion is shown in Fig. 2.1. The equation of motion of the primary mass M, between impacts, is

$$M\ddot{x} + c\dot{x} + kx = F_o \sin \Omega t \quad . \tag{2.1}$$

Following the method suggested by Warburton(4), assume the disturbing force to be $F_o \sin(\Omega t + \alpha)$, where α is an unknown phase angle. Eq. (2.1) now becomes

$$M\ddot{x} + c\dot{x} + kx = F_o \sin(\Omega t + \alpha) \quad . \tag{2.2}$$

The complete solution of Eq. (2.2) is

$$x = e^{-\delta\omega t}(B_1 \sin \eta\omega t + B_2 \cos \eta\omega t) + A \sin(\Omega t + \tau) \tag{2.3}$$

where

$$\delta = c/c_{cr}$$

$$c_{cr} = 2\sqrt{kM}$$

$$\omega = \sqrt{k/M}$$

$$\eta = \sqrt{1 - \delta^2}$$

$$r = \Omega/\omega$$

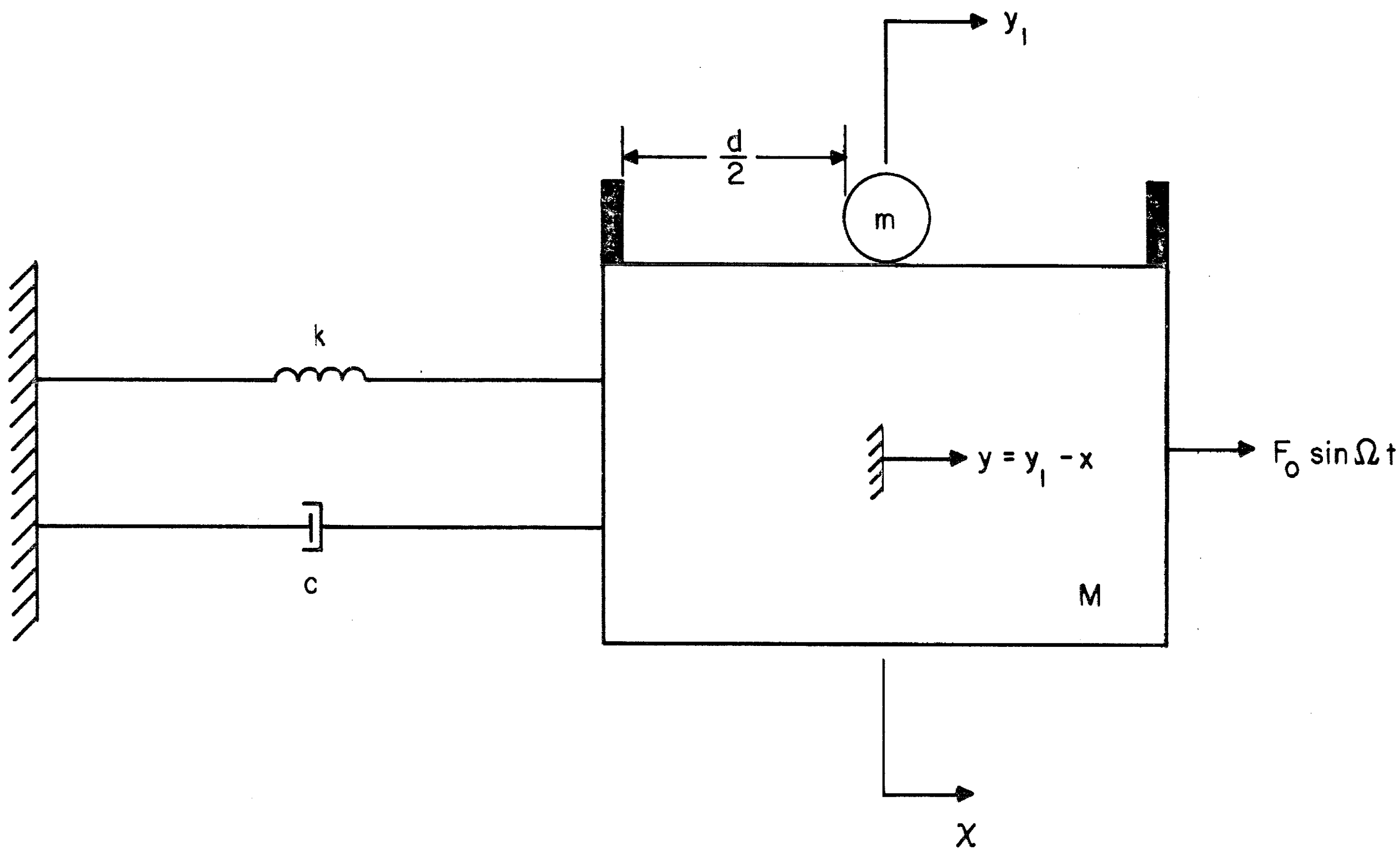


Fig. 2.1. Model of system.

$$A = \frac{F_o/k}{\sqrt{(1-r^2)^2 + (2\delta r)^2}}$$

$$\tau = \alpha - \psi$$

$$\psi = \tan^{-1}\frac{2\delta r}{1-r^2} \qquad 0 < \psi < \pi .$$

For steady state motion with two impacts per cycle, if one impact is assumed to occur at time $t = 0$, the next impact will occur at $t = \frac{\pi}{\Omega}$. Then, as shown in Fig. 2.2, if immediately preceding the impact at $t = 0_-$ (the time immediately preceding or following an impact at time t is denoted by t_- and t_+, respectively), $x = x_b$, $\dot{x} = \dot{x}_b$, $y = \frac{d}{2}$ and the absolute velocity of the particle $V_m = v$. The duration of the impact is very small compared to the natural frequency of the primary system, hence it is reasonable to assume that at $t = 0_+$, the positions of M and m remain the same while their respective absolute velocities are discontinuously changed to $\dot{x}_a$ and $-v$. In order to have two impacts per cycle, at $t = (\frac{\pi}{\Omega})_-$ the displacements and velocities should be the negative of those at $t = 0_-$. Therefore, at $t = (\frac{\pi}{\Omega})_-$, $x = -x_b$, $\dot{x} = -\dot{x}_b$, $y = -\frac{d}{2}$, and $V_m = -v$.

To summarize, the system should satisfy the following conditions:

$$\left.\begin{array}{llllll} \text{at} & t = 0_- & x = x_b & y = \frac{d}{2} & \dot{x} = \dot{x}_b & V_m = v \quad \text{(a)} \\ & t = 0_+ & x = x_b & y = \frac{d}{2} & \dot{x} = \dot{x}_a & V_m = -v \quad \text{(b)} \\ & t = (\frac{\pi}{\Omega})_- & x = -x_b & y = -\frac{d}{2} & \dot{x} = -\dot{x}_b & V_m = -v \quad \text{(c)} \end{array}\right\} \qquad (2.4)$$

Equation (2.3) describes the motion of M from $t = 0_+$ to $t = (\frac{\pi}{\Omega})_-$.

Since the motion of the system during impact must satisfy the momentum equation, then

$$M\dot{x}_- + m V_{m_-} = M\dot{x}_+ + m V_{m_+} , \qquad (2.5)$$

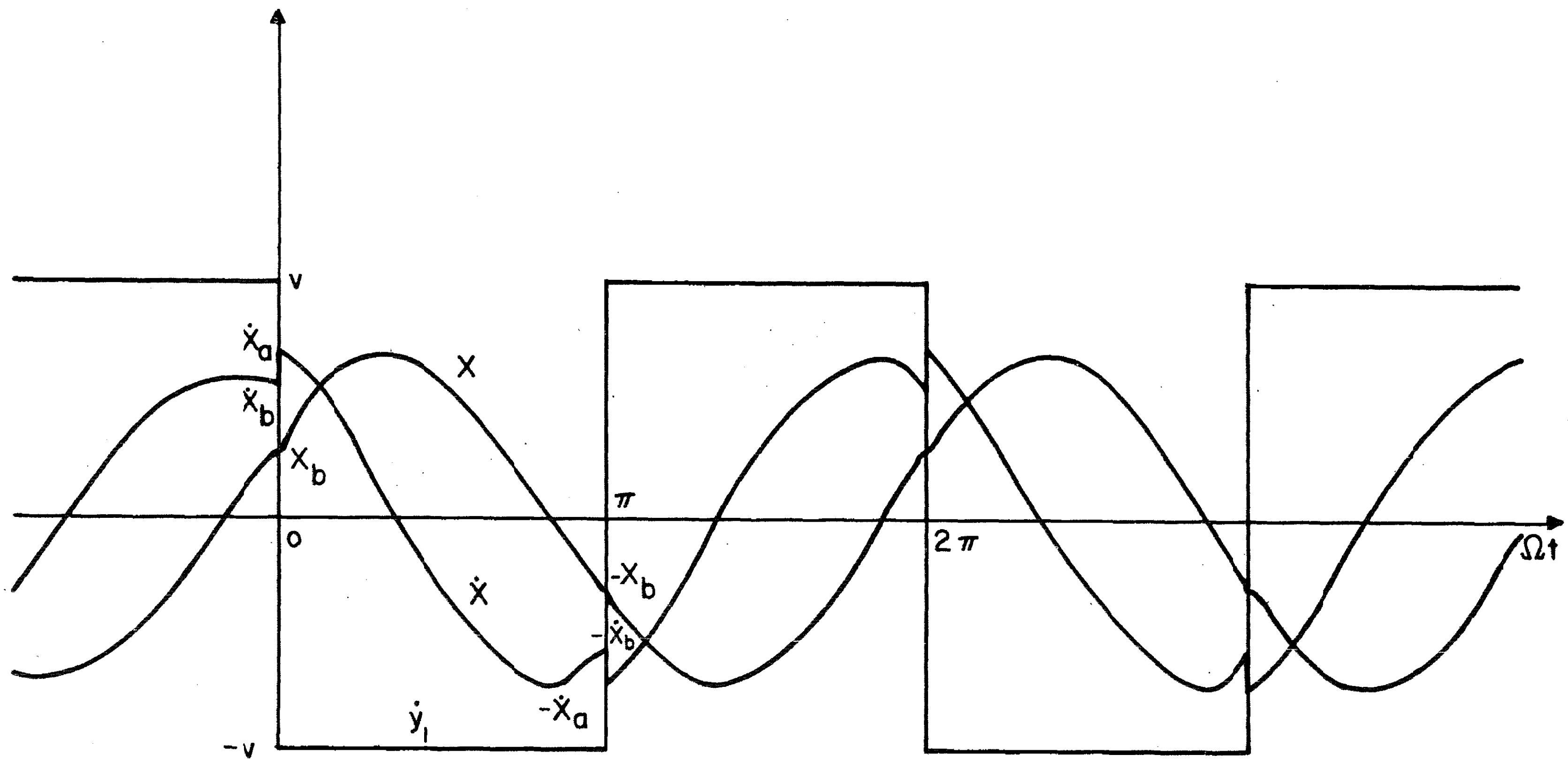


Fig. 2. 2. Wave form of symmetric 2 impacts/cycle motion.

and from the definition of the coefficient of restitution e,[10]

$$\dot{x}_+ - V_{m_+} = -e(\dot{x}_- - V_{m_-}) \; . \tag{2.6}$$

From Eqs. (2.5) and (2.6) the following relations that will be useful later on can be obtained.

$$\dot{x}_+ = \left(\frac{1 - \mu e}{1 + \mu}\right)\dot{x}_- + \frac{\mu(1 + e)}{1 + \mu} V_{m_-} \tag{2.7}$$

$$V_{m_+} = \left(\frac{1 + e}{1 + \mu}\right)\dot{x}_- + \left(\frac{\mu - e}{1 + \mu}\right) V_{m_-} \tag{2.8}$$

$$\dot{x}_- = \frac{(e - \mu)\, V_{m-} + (1 + \mu)\, V_{m+}}{1 + e} \tag{2.9}$$

$$\dot{x}_+ = \frac{e(1 + \mu)\, V_{m-} + (1 - \mu e) V_{m+}}{1 + e} \tag{2.10}$$

where $\mu = \dfrac{m}{M}$.

In steady state motion, the absolute speed of the particle is constant, and it is equal to

$$v = (d + 2x_b)\,\frac{\Omega}{\pi} \; . \tag{2.11}$$

If at $t = 0_-$ the absolute velocity of the particle is $V_{m_-} = v$, then $V_{m_+} = -v$.

Now, if the impact relations (2.9) and (2.10) are evaluated at $t = 0$, since

$$\dot{x}_- = \dot{x}_b \qquad\qquad V_{m_-} = v$$

$$\dot{x}_+ = \dot{x}_a \qquad\qquad V_{m_+} = -v \; ,$$

and by using (2.11), then

$$x_b + \frac{\pi}{2\Omega}\left(\frac{1 + e}{1 - e + 2\mu}\right)\mathring{x}_b = -\frac{d}{2} \tag{2.12}$$

and

$$x_b + \frac{\pi}{2\Omega}\left(\frac{1+e}{1-e-2\mu e}\right)\dot{x}_a = -\frac{d}{2} \quad (2.13)$$

An expression describing the velocity of M can be obtained by differentiating Eq. (2.3) with respect to t. Thus

$$\begin{aligned}\dot{x} = &- \delta\omega e^{-\delta\omega t}(B_1 \sin \eta\omega t + B_2 \cos \eta\omega t) \\ &+ e^{-\delta\omega t}(B_1\, \eta\omega \cos \eta\omega t - B_2\, \eta\omega \sin \eta\omega t) \\ &- A\Omega \cos (\Omega t + \alpha - \psi) \quad .\end{aligned} \quad (2.14)$$

Equations (2.3) and (2.14) describe the displacement and velocity of M during the time interval $0_+ \le t \le (\frac{\pi}{\Omega})_-$. From Eqs. (2.3), (2.4-b), and (2.4-c)

$$x(0_+) = x_b = B_2 + A \sin \tau \quad (2.15)$$

$$x\left((\tfrac{\pi}{\Omega})_-\right) = -x_b = e^{-\frac{\delta\pi}{r}}(B_1 \sin \eta\tfrac{\pi}{r} + B_2 \cos \eta\tfrac{\pi}{r}) - A \sin \tau \quad . \quad (2.16)$$

Similarly, from Eqs. (2.14), (2.4-b), and (2.4-c)

$$\dot{x}(0_+) = \dot{x}_a = -\delta\omega B_2 + B_1\, \eta\omega + A\Omega \cos \tau \quad (2.17)$$

$$\begin{aligned}\dot{x}\left((\tfrac{\pi}{\Omega})_-\right) = -\dot{x}_b = &-\delta\omega e^{-\frac{\delta\pi}{r}}(B_1 \sin \eta\tfrac{\pi}{r} + B_2 \cos \eta\tfrac{\pi}{r}) \\ &+ \eta\omega e^{-\frac{\delta\pi}{r}}(B_1 \cos \eta\tfrac{\pi}{r} - B_2 \sin \eta\tfrac{\pi}{r}) \\ &- A\Omega \cos \tau \quad .\end{aligned} \quad (2.18)$$

By using the following new variables,

$$S \equiv \sin \tau$$

$$C \equiv \Omega \cos \tau$$

$$h_1 \equiv e^{-\frac{\delta\pi}{r}} \sin \eta \frac{\pi}{r}$$

$$h_2 \equiv e^{-\frac{\delta\pi}{r}} \cos \eta \frac{\pi}{r}$$

$$\sigma_1 \equiv \frac{\pi}{2\Omega} \frac{1 + e}{1 - e + 2\mu}$$

$$\sigma_2 \equiv \frac{\pi}{2\Omega} \frac{1 + e}{1 - e - 2\mu e}$$

$$\theta_1 \equiv \omega e^{-\frac{\delta\pi}{r}} \left[- \delta \sin \eta \frac{\pi}{r} + \eta \cos \eta \frac{\pi}{r} \right]$$

$$\theta_2 \equiv \omega e^{-\frac{\delta\pi}{r}} \left[- \delta \cos \eta \frac{\pi}{r} - \eta \sin \eta \frac{\pi}{r} \right] ,$$

Eqs. (2. 15), (2. 18), (2. 16), (2. 17), (2. 12), and (2. 13) can be expressed respectively as:

$$\left.\begin{aligned}
& x_b - B_2 - SA = 0 \\
& \mathring{x}_b + \theta_1 B_1 + \theta_2 B_2 - CA = 0 \\
& x_b + h_1 B_1 + h_2 B_2 - SA = 0 \\
& \dot{x}_a - \eta\omega B_1 + \delta\omega B_2 - CA = 0 \\
& x_b + \sigma_1 \mathring{x}_b = -\frac{d}{2} \\
& x_b + \sigma_2 \dot{x}_a = -\frac{d}{2} .
\end{aligned}\right\} \qquad (2.19)$$

Eq. (2. 19) furnishes 6 relations among the 6 unknowns α, B_1, B_2, x_b, $\dot{x}_b$, and $\dot{x}_a$.

In matrix notation, Eq. (2. 19) can be put in the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & -S \\ 0 & 0 & 1 & -\eta\omega & \delta\omega & -C \\ 1 & 0 & 0 & h_1 & h_2 & -S \\ 0 & 1 & 0 & \theta_1 & \theta_2 & -C \\ 1 & \sigma_1 & 0 & 0 & 0 & 0 \\ 1 & 0 & \sigma_2 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_b \\ \dot{x}_b \\ \dot{x}_a \\ B_1 \\ B_2 \\ A \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\frac{d}{2} \\ -\frac{d}{2} \end{Bmatrix} . \tag{2.20}$$

From the solution of (2.20), expressions for A, B_1, and B_2 in terms of the known parameters can be obtained. Thus

$$A = \frac{N(A)}{\Delta} \tag{2.21}$$

$$B_1 = \frac{N(B_1)}{\Delta} \tag{2.22}$$

$$B_2 = \frac{N(B_2)}{\Delta} \tag{2.23}$$

where

$$N(A) \equiv \frac{d}{2}\left[h_1(\sigma_1\theta_2 - \sigma_2\omega\delta) - (\sigma_1\theta_1 + \eta\sigma_2\omega)(1+h_2) \right]$$

$$N(B_1) \equiv \frac{d}{2}(1+h_2)(\sigma_2 - \sigma_1)\ C$$

$$N(B_2) \equiv \frac{d}{2} h_1(\sigma_1 - \sigma_2)\ C$$

$$\Delta \equiv h_1\left[C(\sigma_2 - \sigma_1) - (S + C\sigma_2)\sigma_1\theta_1 + (S + C\sigma_1)\delta\omega\sigma_2 \right]$$

$$+ (1+h_2)\left[(S + C\sigma_2)\sigma_1\theta_1 + (S + C\sigma_1)\eta\omega\sigma_2 \right] .$$

If the damping in the primary system is zero, Eq. (2.21) reduces to Eq. (9) in Ref. 4.

Eq. (2.21) can be put in the form

$$2 \sin \tau + H \cos \tau = -\rho \tag{2.24}$$

where $\rho = \frac{d}{A}$

$$H = 2\Omega \left\{ \frac{\left[(\sigma_2 - \sigma_1) + \sigma_1\sigma_2(\delta\omega - \theta_2)\right]h_1 + \left[\sigma_1\sigma_2(\theta_1 + \eta\omega)\right](1 + h_2)}{\left[\delta\sigma_2\omega - \sigma_1\theta_2\right]h_1 + \left[\sigma_1\theta_1 + \eta\sigma_2\omega\right](1 + h_2)} \right\}.$$

Solution of Eq. (2.24) for τ results in:

$$\sin \tau = \frac{-2\rho \pm H\sqrt{H^2 + 4 - \rho^2}}{H^2 + 4}$$

$$\cos \tau = \frac{-\rho H \mp H\sqrt{H^2 + 4 - \rho^2}}{H^2 + 4}$$

$$\therefore \qquad \tau = \tan^{-1}\left[\frac{-2\rho \pm H\sqrt{H^2 + 4 - \rho^2}}{-\rho H \mp 2\sqrt{H^2 + 4 - \rho^2}}\right]. \tag{2.25}$$

In order to have real values for $\sin \tau$ and $\cos \tau$, the clearance d cannot be arbitrarily large; it should satisfy the relation $\rho^2 \leq H^2 + 4$. The physical interpretation of this restriction is that for d exceeding this limit, the actual system will not have a two-impacts-per-cycle steady state motion.

The two sets of signs appearing in Eq. (2.25) correspond to two distinct steady state solutions. Grubin[2], who encountered a similar situation, chose the set of signs that resulted in a steady state solution that agreed with the numerical solution of the equation of motion of the system.

Since the conditions that were used to obtain Eq. (2.25) are the exact conditions that must be met by the system if it is in steady state motion with two impacts per cycle, then, as seen from Eq. (2.25), there are two possible steady state solutions. The analytical criteria of deciding which solution will be valid, if any, will be furnished by the stability analysis of the solutions. Such an analysis is carried out in

Chapter 3.

With the value of τ determined from (2.25), B_1 and B_2 can be found from (2.22) and (2.23). Since

$$x = e^{-\delta\omega t}(B_1 \sin \eta\omega t + B_2 \cos \eta\omega t) + A \sin(\Omega t + \tau), \qquad (2.26)$$

then the motion of the primary system is determined.

For the special case of resonance and no damping, the steady state solution is considerably simpler, and it will be obtained for use later on.

First, let $\delta = 0$, then

$$x = B_1 \sin \omega t + B_2 \cos \omega t + A \sin(\Omega t + \tau) \qquad (2.27)$$

and

$$\dot{x} = B_1 \omega \cos \omega t - B_2 \omega \sin \omega t + A\Omega \cos(\Omega t + \tau) \,. \qquad (2.28)$$

Since $x(0+) = x_b$ and $\dot{x}(0+) = \dot{x}_a$,

$$\therefore \qquad B_2 = x_b - A \sin \tau \qquad (2.29)$$

$$B_1 = \frac{\dot{x}_a}{\omega} - Ar \cos \tau \,. \qquad (2.30)$$

Let $r = 1 + \epsilon$, $\epsilon << 1$. Then

$$\sin(\Omega t + \tau) = \sin\left[\epsilon\omega t + (\omega t + \tau)\right]$$

$$= \epsilon\omega t \cos(\omega t + \tau) + 0(\epsilon^2) \,. \qquad (2.31)$$

Substituting (2.29) and (2.30) in (2.27), and making use of (2.31),

$$x = x_b \cos \omega t + \frac{\dot{x}_a}{\omega} \sin \omega t + A\epsilon\Big[\omega t \cos(\omega t + \tau) - \cos \tau \sin \omega t\Big] \,. \qquad (2.32)$$

Since

$$\lim_{\varepsilon \to 0} A\varepsilon = \lim_{\varepsilon \to 0} \frac{\frac{F_o}{k}\varepsilon}{-2\varepsilon} = -\frac{F_o}{2k} ,$$

then

$$x = x_b \cos \omega t + \frac{\dot{x}_a}{\omega} \sin \omega t - \frac{F_o}{2k}\left[\omega t \cos(\omega t + \tau) - \cos\tau \sin \omega t\right] . \quad (2.33)$$

Using the condition that $x\left(\left(\frac{\pi}{\omega}\right)_-\right) = -x_b$ in conjunction with (2.33), τ is found to be either $\pm\frac{\pi}{2}$. Consequently, Eq. (2.33) simplifies to

$$x = x_b \cos \omega t + \frac{\dot{x}_a}{\omega} \sin \omega t - \frac{F_o}{2k} \omega t \cos(\omega t + \tau) . \quad (2.34)$$

From the condition that $\dot{x}\left(\left(\frac{\pi}{\omega}\right)_-\right) = -\dot{x}_b$, one obtains by differentiating Eq. (2.34) and substituting in it $\omega t = \pi$, that

$$\dot{x}_a - \dot{x}_b = \frac{F_o}{2k} \pi\omega \sin(\pi + \tau) . \quad (2.35)$$

But, by subtracting (2.12) from (2.13) and rearranging terms

$$\dot{x}_a - \dot{x}_b = 2\frac{\mu\omega}{\pi}(d + 2x_b) \quad (2.36)$$

and since all the terms on the right hand side of (2.36) are positive, then by comparing (2.35) to (2.36) it is seen that τ should be $-\frac{\pi}{2}$, and that

$$(d + 2x_b) = \frac{\pi^2}{4\mu}\frac{F_o}{k} . \quad (2.37)$$

Making use of (2.13) and (2.37) one obtains

$$x_b = \frac{\pi^2}{8\mu}\frac{F_o}{k} - \frac{d}{2} , \quad (2.38)$$

and

$$\frac{\dot{x}_a}{\omega} = -\left(\frac{1 - e - 2\mu e}{1 + e}\right)\frac{\pi}{4\mu}\frac{F_o}{k} \quad (2.39)$$

Substituting (2. 38), (2. 39) and $\tau = -\frac{\pi}{2}$ in (2. 34),

$$x = \left(\frac{\pi^2}{8\mu}\ \frac{F_o}{k} - \frac{d}{2}\right) \cos \omega t - \left(\frac{1 - e - 2\mu e}{1 + e}\right) \frac{\pi}{4\mu}\ \frac{F_o}{k} \sin \omega t$$

$$- \frac{F_o}{2k}\ \omega t \sin \omega t \ , \qquad (2.40)$$

$$0 \le \omega t \le \pi \ .$$

2. 3. Energy Balance

The concept of energy balance can be used in this case to furnish a measure of the amount of work "dissipated" per impact and through the mechanism of viscous damping. In steady state 2 impacts/cycle motion, the energy input should equal the energy output, during the time interval $\Omega t = 0$ to $\Omega t = \pi$, i. e.

$$w_{in} = w_d + w_{imp} \qquad (2.41)$$

where w_{in} = work done by $F_o \sin(\Omega t + \alpha)$

w_d = work dissipated by damping

w_{imp} = kinetic energy lost per impact .

Equation (2. 41) can be expressed as[11]:

$$\frac{1}{\omega} \int_0^{\frac{\pi}{r}} F_o \sin(\Omega t + \alpha)\, \dot{x}\, d\theta = (c\omega) \int_0^{\frac{\pi}{r}} \left(\frac{\dot{x}}{\omega}\right)^2 d\theta + w_{imp} \qquad (2.42)$$

where $\theta \equiv \omega t$.

By making use of Eq. (2. 26), and carrying out the integration, it is found that

$$w_{in} = \Bigg[\frac{1}{2}\phi_1 F_o \Big\{ \phi_5 \big[-\delta\cos(\phi_3+\psi) + (\eta+r)\sin(\phi_3+\psi) \big] - \phi_6 \big[-\delta\cos(\phi_4-\psi) + (\eta-r)\sin(\phi_4-\psi) \big] \Big\} e^{-\delta\theta}$$

$$+ \frac{1}{2}\phi_2 F_o \Big\{ \phi_5 \big[-\delta\sin(\phi_3+\psi) - (\eta+r)\cos(\phi_3+\psi) \big] - \phi_6 \big[-\delta\sin(\phi_4-\psi) - (\eta-r)\cos(\phi_4-\psi) \big] \Big\} e^{-\delta\theta}$$

$$+ \frac{1}{2} Ar F_o \Big\{ \theta\sin\psi - \frac{1}{2r}\cos\big[2(r\theta+\tau) + \psi \big] \Big\} \Bigg]_{\theta=0}^{\theta=\frac{\pi}{r}}$$

$$\frac{w_d}{c\omega} = \Bigg[-\frac{\phi_1^2}{4} e^{-2\delta\theta} \Big[2\sin\eta\theta(\delta\sin\eta\theta + \eta\cos\eta\theta) + \frac{\eta^2}{\delta} \Big]$$

$$- \frac{\phi_2^2}{4} e^{-2\delta\theta} \Big[2\cos\eta\theta(\delta\cos\eta\theta - \eta\sin\eta\theta) + \frac{\eta^2}{\delta} \Big]$$

$$+ A^2 r^2 \Big[\frac{\theta}{2} + \frac{1}{4r}\sin(2r\theta + 2\tau) \Big]$$

$$+ \frac{\phi_1\phi_2}{2} e^{-2\delta\theta} \Big[\delta\sin 2\eta\theta + \eta\cos 2\eta\theta \Big]$$

$$- \phi_1 Ar \Big\{ \phi_5 \big[-\delta\sin\phi_3 - (\eta+r)\cos\phi_3 \big] + \phi_6 \big[-\delta\sin\phi_4 - (\eta-r)\cos\phi_4 \big] \Big\} e^{-\delta\theta}$$

$$+ \phi_2 Ar \Big\{ \phi_5 \big[-\delta\cos\phi_3 + (\eta+r)\cos\phi_3 \big] + \phi_6 \big[-\delta\cos\phi_4 + (\eta-r)\sin\phi_4 \big] \Big\} e^{-\delta\theta} \Bigg]_{\theta=0}^{\theta=\frac{\pi}{r}}$$

where
$$\phi_1 \equiv \delta B_1 + \eta B_2$$
$$\phi_2 \equiv \eta B_1 - \delta B_2$$
$$\phi_3 \equiv (\eta + r)\theta + \tau$$
$$\phi_4 \equiv (\eta - r)\theta - \tau$$
$$\phi_5 \equiv \frac{1}{\delta^2 + (\eta + r)^2}$$
$$\phi_6 \equiv \frac{1}{\delta^2 + (\eta - r)^2} \quad .$$

From (2. 5) and (2. 6),

$$w_{imp} - 2\mu(1-e)\left(\frac{1+\mu}{1+e}\right)\left[\frac{\Omega}{\pi}(d+2x_b)\right]^2 M \quad .$$

It was found that (2. 42) holds true even when thc solution is unstable.

3. STABILITY

3.1. Theoretical Considerations

Now that we have a particular steady state solution for the system under consideration, we can proceed to investigate the stability of this solution. The type of stability that we are concerned with in this case is asymptotic stability.

Let the differential equation of motion of our system be expressed in the form

$$\dot{\vec{Z}} = \vec{F}(Z_1, \ldots, Z_4, t) \quad , \qquad (3.1)$$

and let a particular solution of (3.1) be

$$\vec{Z} - \vec{S}(t) \quad . \qquad (3.2)$$

If this solution is perturbed slightly, so that

$$\vec{Z}_p = \vec{S}(t) + \vec{\xi}(t) \ , \qquad (3.3)$$

the solution is said to be asymptotically stable if

$$\lim_{t \to \infty} |\xi_i(t)| = 0 \qquad \text{for } i = 1, \ldots, 4 \ .$$

In the usual cases, the time behavior of $\vec{\xi}(t)$ can be furnished by the so-called variational equations,[12]

$$\dot{\vec{\xi}} = M \vec{\xi} \qquad (3.4)$$

where

$$M_{ij} = \frac{\partial F_i(\vec{S}, t)}{\partial S_j} \quad , \qquad (i, j = 1, \ldots, 4) \ .$$

Assuming that the ξ's are of the form λ^t, then if the modulus of all the roots of the characteristic equation of (3.4) is less or greater than unity, the solution is correspondingly asymptotically stable or unstable.

For the present case, since the 2 impacts/cycle solution is valid only for the range $0_+ \le \Omega t \le \pi_-$, the above-mentioned matrix M cannot be determined by this classical method.

By borrowing ideas from error propagation in difference equations(13), we perturb our solution immediately after one impact, and then determine the deviation of the resulting solution from steady state conditions immediately after the following impact. By repeating this process over and over again, we can determine the propagation (similar to change with time) of the initial perturbations in the solution. The stability or instability of the solution is determined by whether or not the deviations from the steady state solution decay or grow, as the number of impacts is increased indefinitely (i. e. , as $t \to \infty$).

The differential equation of motion of our system, between impacts,

$$\ddot{x} = - \omega^2 x - 2\delta\omega\dot{x} + \frac{F_o}{M} \sin \Omega t$$

$$\ddot{y}_1 = 0$$

can be put in the form(14)

$$\frac{d\vec{x}}{dt} = \vec{f}(\vec{x}, t) \tag{3.5}$$

where

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \qquad \vec{f}(\vec{x}, t) = \begin{pmatrix} f_1(\vec{x}, t) \\ f_2(\vec{x}, t) \\ f_3(\vec{x}, t) \\ f_4(\vec{x}, t) \end{pmatrix};$$

and

$$x_1 = x \qquad f_1 = x_2$$

$$x_2 = \dot{x} \qquad f_2 = -\omega^2 x_1 - 2\delta\omega x_2 + \frac{F_o}{M}\sin\Omega t$$

$$x_3 = y_1 \qquad f_3 = x_4$$

$$x_4 = \dot{y}_1 \qquad f_4 = 0 \quad .$$

Since the first partial derivatives of the 4 functions $f_i(\vec{x}, t)$, $i = 1, \ldots, 4$ with respect to their 5 variables, $x_1, \ldots, x_4$, and t, exist and are continuous, then each of the functions $f_i(\vec{x}, t)$ satisfies a Lipschitz condition.(15)

Furthermore, if the initial conditions are specified, the Cauchy-Lipschitz theorem states that the solution of (3.5) exists and is unique in $E_4 \times t$ space.

This type of motion can be represented in the phase plane by a periodic process (limit cycle) as shown in Fig. 3.1. On the analytic trajectories AB and CD, the motion of the system is governed by Eq. (3.5). On the stretches BC and DA, where the small impact time is idealized to be infinitesimally small, Eq. (3.5) does not apply, but the motion of the system is determined by the invariants of the problem--the impact conditions, Eqs. (2.5) and (2.6). These invariants relate the conditions at C and A to those at B and D, respectively.

Now let the solution curve be perturbed slightly right after an impact, e.g. at A. By means of the Cauchy-Lipschitz theorem and the impact conditions, the perturbations at point A are continuously related to the perturbations at point C.

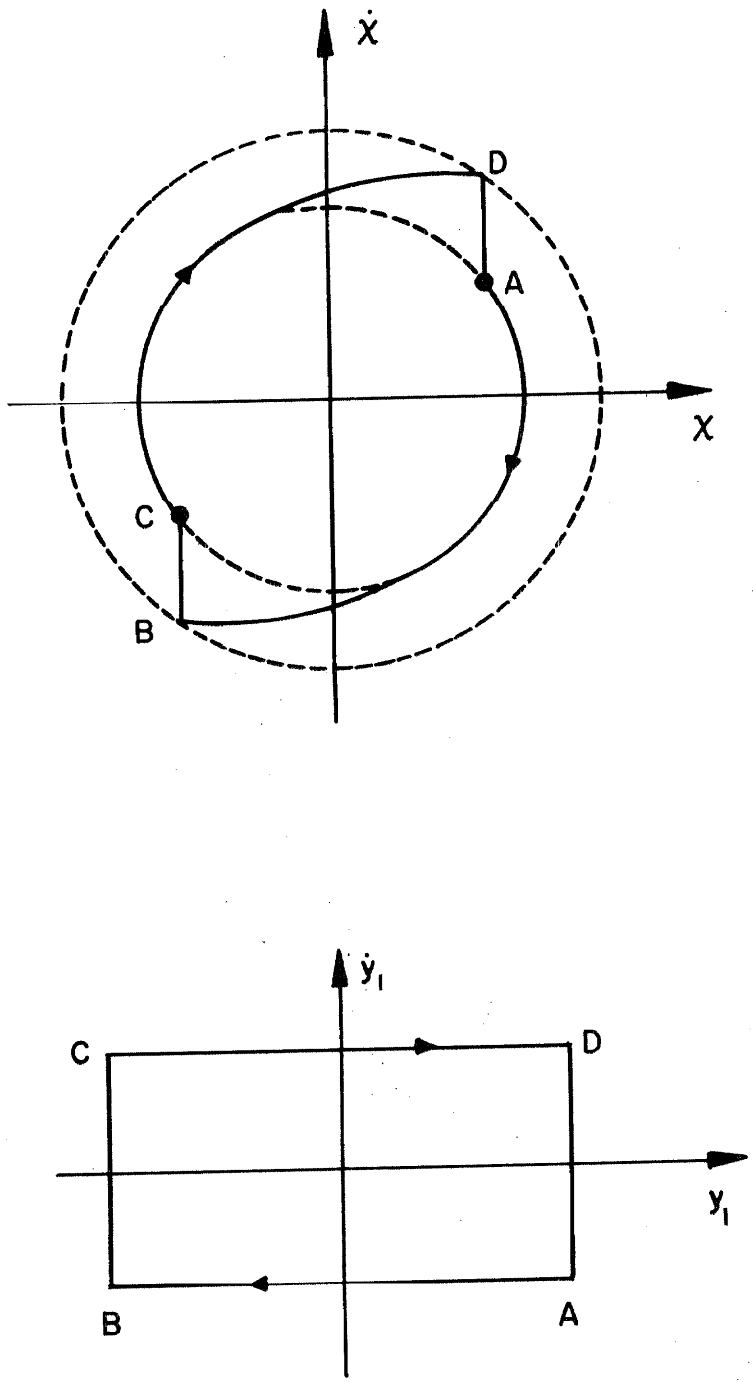


Fig. 3.1. Phase plane representation of periodic 2 impacts/cycle motion.

If at $\Omega t = (\Delta t_o)_+$ the steady state solution is perturbed by a small amount $\vec{\xi}_{(0)}$, then the time of the next impact, $\Omega t = \pi + \Delta t'_o$, is determined by a relation of the form

$$\Delta t'_o = g(\vec{S}, \vec{\xi}_{(0)}, \Delta t'_o)$$

or

$$G(\vec{S}, \vec{\xi}_{(0)}, \Delta t'_o) = 0 \quad . \tag{3.6}$$

From the theory of implicit functions, it is known that[16] if G is a continuous function of its arguments, and if at the point $(\vec{S}^*, \vec{\xi}^*_{(0)}, \Delta t^{*\prime}_o)$

i) $G = 0$

ii) G is differentiable

iii) $\dfrac{\partial G}{\partial \Delta t'_o} \neq 0$

then there exists at least one function $\Delta t'_o = \Delta t'_o(\vec{S}, \vec{\xi}_{(0)})$ reducing to $\Delta t^{*\prime}_o$ at the point $(\vec{S}^*, \vec{\xi}^*_{(0)})$, and which, in the neighborhood of this point, satisfies the equation $G(\vec{S}, \vec{\xi}_{(0)}, \Delta t'_o) = 0$ identically.

Furthermore, if in this neighborhood $\dfrac{\partial G}{\partial \Delta t'_o}$ exists and is not zero, the solution of the equation $G = 0$ is unique.

By making use of this theorem, at $\Omega t = (\pi + \Delta t'_o)_+$ the deviation of the solution from steady state can be put in the form

$$\vec{\xi}_{(1)} = P\,\vec{\xi}_{(0)} + \vec{R}\,(\vec{\xi}_{(0)})$$

where P is a constant matrix, and $\vec{R}$ contains all terms of ξ_i higher than the first power.

Since the 2 impacts/cycle solution repeats itself after intervals of $\Omega t = \pi$, the perturbation at $\Omega t = (2\pi + \Delta t''_o)_+$ will be

$$\vec{\xi}_{(2)} = P\vec{\xi}_{(1)} + \vec{R}(\vec{\xi}_{(1)}) \ .$$

By following the perturbed solution from one impact to the next one, we obtain the continuous transformation

$$\vec{\xi}_{(n+1)} = P\vec{\xi}_{(n)} + \vec{R}(\vec{\xi}_{(n)}) \tag{3.7}$$

$$= P^{n+1}\vec{\xi}_{(0)} + \vec{R}'(\vec{\xi}_{(0)}) \tag{3.8}$$

It is worth noting that if the sign of P did change (it does not in this case) after each impact, the net effect would be to multiply P in (3.7) by $(-1)^{n+1}$. This has no effect on the stability criteria, which, as will be proved below, depends on the modulus of the eigenvalues of P.

Consider the linear part of Eq. (3.8), i.e.

$$\vec{\xi}_{(n+1)} = P^{n+1}\vec{\xi}_{(0)} \ . \tag{3.9}$$

Equation (3.9) will be asymptotically stable if and only if $\lim_{n\to\infty} P^n = 0$.

Now if the eigenvalues of P are distinct, it can be diagonalized by a similarity transformation[17], so that

$$P^n = S\,\Lambda^n\,S^{-1} \ .$$

The requirement that

$$\lim_{n\to\infty} P^n = 0$$

is satisfied if and only if all the eigenvalues of P have modulus less than unity, i.e. if

$$|\lambda_i| < 1 \quad , \qquad (i = 1, \ldots, 4) \ .$$

Actually, the requirement that the eigenvalues be distinct is superfluous, since if they are not, P can be transformed into the Jordan canonical form[18]. Then the same results as above can be obtained, since all we need is that

$$\lim_{n\to\infty} \lambda_i^n = 0 \qquad \text{for } i = 1, \ldots, 4 .$$

Theorem: If

i) $\|\vec{\xi}_{(0)}\|$ is sufficiently small,

ii) $$\lim_{\|\vec{\xi}\|\to 0} \frac{\|\vec{R}(\vec{\xi})\|}{\|\vec{\xi}\|} = 0 ,$$

and if

$$\vec{\xi}_{(n+1)} = P^{n+1} \vec{\xi}_{(0)} \tag{3.10}$$

is asymptotically stable, then also so is

$$\vec{\xi}_{(n+1)} = P^{n+1} \vec{\xi}_{(0)} + \vec{R}(\vec{\xi}_{(0)}) . \tag{3.11}$$

Proof:

An alternative form of (3.11) is

$$\vec{\xi}_{(n)} = P^n \vec{\xi}_{(0)} + \sum_{i=0}^{n-1} P^{n-1-i} \vec{R}(\vec{\xi}_{(i)}) , \qquad n \geq 1 .$$

By virtue of (3.10) being asymptotically stable, then

$$\| P^n \| \leq C e^{-\alpha n}$$

with $C > 1$ and $\alpha > 0$.

$$\therefore \qquad \| \vec{\xi}_{(n)} \| \leq C e^{-\alpha n} \| \vec{\xi}_{(0)} \| + \sum_{i=0}^{n-1} C e^{-\alpha(n-1-i)} \| \vec{R}(\vec{\xi}_{(i)}) \| .$$

Making use of the initial assumptions, therefore there exists a constant δ^* such that if $\|\vec{\xi}\| < \delta^*$, then

$$\| \vec{R}(\vec{\xi}) \| \leq \frac{\alpha}{2C} \|\vec{\xi}\| e^{-\alpha} ;$$

hence

$$e^{\alpha n}\,\|\vec{\xi}_{(n)}\| \leq C\|\vec{\xi}_{(0)}\| + \frac{\alpha}{2}\sum_{i=0}^{n-1} e^{\alpha i}\|\vec{\xi}_{(i)}\| \quad . \tag{3.12}$$

Let $S_i = e^{\alpha i}\,\|\vec{\xi}_{(i)}\|$ and $\delta^* = C\,\|\vec{\xi}_{(0)}\|$. Then from (3.12)

$$S_n \leq \delta^* + \frac{\alpha}{2}\sum_{i=0}^{n-1} S_i$$

$$\therefore \qquad \begin{aligned} S_1 &\leq \delta^* + \frac{\alpha}{2} S_0 \\ &\leq \delta^*(1+\frac{\alpha}{2}) \\ &\leq \delta^* e^{\frac{\alpha}{2}} \end{aligned}$$

and

$$\begin{aligned} S_2 &\leq \delta^* + \frac{\alpha}{2}(S_0+S_1) \\ &\leq \delta^*\left[1 + 2(\frac{\alpha}{2}) + (\frac{\alpha}{2})^2\right] \\ &\leq \delta^* e^{2\cdot\frac{\alpha}{2}} \quad . \end{aligned}$$

Similarly

$$S_n \leq \delta^* e^{n\frac{\alpha}{2}} \quad .$$

Hence

$$e^{\alpha n}\,\|\vec{\xi}_{(n)}\| \leq \delta^* e^{n\frac{\alpha}{2}}$$

$$\|\vec{\xi}_{(n)}\| \leq \delta^* e^{-\frac{\alpha n}{2}}$$

$$\therefore \qquad \lim_{n\to\infty}\|\vec{\xi}_{(n)}\| = 0 \quad . \qquad\qquad \text{Q.E.D.}$$

Thus, our problem is to determine P and to examine its eigenvalues.

Since our system has two degrees of freedom, $\vec{\xi}$ should be a 4-component vector. A proper choice of components can be the two displacements, and the two velocities. However, it is more natural for this system if the phase angle between the motion of the particle and M is used as one of the components, instead of the displacement of the particle. In other words, the initial perturbation will consist of small variation of the steady state values of x, $\dot{x}$, v, and τ.

The conditions that we have are:

Steady State	Perturbed
At $\Omega t = 0_+$,	At $\Omega t = (0 + \Delta t_o)_+$,
$x = x_o$	$x = x_o + \Delta x_o$
$y = \frac{d}{2}$	$y = \frac{d}{2}$
$\dot{x} = \dot{x}_o$	$\dot{x} = \dot{x}_o + \Delta \dot{x}_o$
$v = -v_o$	$v = -(v_o + \Delta v_o)$
$\tau = \tau_o$	$\tau = \tau_o + \Delta \tau_o$
At $\Omega t = \pi_+$,	At $\Omega t = (\pi + \Delta t_o')_+$,
$x = -x_o$	$x = -(x_o + \Delta x_o')$
$y = -\frac{d}{2}$	$y = -\frac{d}{2}$
$\dot{x} = -\dot{x}_o$	$\dot{x} = -(\dot{x}_o + \Delta \dot{x}_o')$
$v = v_o$	$v = v_o + \Delta v_o'$
$\tau = \tau_o$	$\tau = \tau_o + \Delta \tau_o'$.

3.2. Determination of P

Without any loss of generality, ω and $\frac{F_o}{k}$ can each be taken as unity. Then for the general case $(\Omega \neq 1, \delta \neq 0)$

$$x = e^{-\delta t}(B_1 \sin \eta t + B_2 \cos \eta t) + A \sin (\Omega t + \tau) \quad , \tag{3.13}$$

$$\dot{x} = e^{-\delta t}(-\delta \sin \eta t + \eta \cos \eta t)B_1 + e^{-\delta t}(-\delta \cos \eta t - \eta \sin \eta t)B_2 \tag{3.14}$$
$$+ A\Omega \cos (\Omega t + \tau) \quad .$$

Now

$$x(0_+) = x_o = B_{2_o} + A \sin \tau_o \tag{3.15}$$

and

$$\dot{x}(0_+) = \dot{x}_o = \eta B_{1_o} - \delta B_{2_o} + A\Omega \cos \tau_o \tag{3.16}$$

where the o subscript refers to the unperturbed conditions (i. e., $x_o = x_a$, $\dot{x}_o = \dot{x}_a$, etc.).

From (3. 15) and (3. 16)

$$B_{2_o} = x_o - A \sin \tau_o \tag{3.17}$$

and

$$B_{1_o} = \frac{1}{\eta}\left\{\dot{x}_o + \delta B_{2_o} - A\Omega \cos \tau_o\right\} \quad . \tag{3.18}$$

In finding the perturbed values of B_{1_o} and B_{2_o} the quantities with o subscript in the above equations should be replaced by their perturbed values. Thus

$$B_2(0 + \Delta t_o) = (x_o + \Delta x_o) - A \sin (\tau_o + \Delta\tau_o) \ , \tag{3.19}$$

and

$$B_1(0 + \Delta t_o) = \frac{1}{\eta}\left\{\dot{x}_o + \Delta\dot{x}_o + \delta B_2(0 + \Delta t_o) - A\Omega \cos (\tau_o + \Delta\tau_o)\right\} \quad . \tag{3.20}$$

Since $\Delta\tau_o$ is a small quantity of order ϵ, then to first order approximation,

$$B_2(0+\Delta t_o) = B_{2_o} + \Delta x_o - A\Delta\tau_o \cos \tau_o \tag{3.21}$$

and

$$B_1(0+\Delta t_o) = B_{1_o} + \frac{\delta}{\eta}\Delta x_o + \frac{1}{\eta}\Delta\dot{x}_o + a\Delta\tau_o \tag{3.22}$$

where $\quad a \equiv \frac{A}{\eta}(\Omega \sin \tau_o - \delta \cos \tau_o)$.

Equation (3.13) describes the motion of the primary system immediately after $t = \frac{0+\Delta t_o}{\Omega}$ to immediately prior to $t = \frac{\pi + \Delta t'_o}{\Omega}$. Thus the time during which Eq. (3.13) is applicable is $\Omega t = \pi + \Delta T$ where $\Delta T = (\Delta t'_o - \Delta t_o)$. Hence from the condition that $x\left(\frac{\pi+\Delta t'_o}{\Omega}\right) = -(x_o + \Delta x'_o)$, we obtain from Eq. (3.13),

$$-(x_o + \Delta x'_o) = e^{-\delta\left(\frac{\pi+\Delta T}{\Omega}\right)}\left\{B_1(0+\Delta t_o)\sin\eta\left(\frac{\pi+\Delta T}{\Omega}\right) + B_2(0+\Delta t_o)\cos\eta\left(\frac{\pi+\Delta T}{\Omega}\right)\right\} + A\sin(\pi + \Delta T + \tau_o + \Delta\tau_o) \ .$$

$$= C_o + C_1\Delta x_o + C_2\Delta\dot{x}_o + C_3\Delta T + C_4\Delta\tau_o \tag{3.23}$$

where

$$C_o \equiv e^{-\frac{\delta\pi}{\Omega}}\left[B_{1_o}\sin\frac{\eta\pi}{\Omega} + B_{2_o}\cos\frac{\eta\pi}{\Omega}\right] - A\sin\tau_o$$

$$C_1 \equiv e^{-\frac{\delta\pi}{\Omega}}\left[\frac{\delta}{\eta}\sin\frac{\eta\pi}{\Omega} + \cos\frac{\eta\pi}{\Omega}\right]$$

$$C_2 \equiv e^{-\frac{\delta\pi}{\Omega}}\left[\frac{1}{\eta}\sin\frac{\eta\pi}{\Omega}\right]$$

$$C_3 \equiv b_3 + e^{-\frac{\delta\pi}{\Omega}}(b_1 + b_2) - A\cos\tau_o$$

$$C_4 \equiv e^{-\frac{\delta\pi}{\Omega}}\left[a \sin\frac{\eta\pi}{\Omega} - (A\cos\tau_o)\cos\frac{\eta\pi}{\Omega}\right] - A\cos\tau_o$$

$$b_1 \equiv \left(\frac{\eta}{\Omega}\cos\frac{\eta\pi}{\Omega}\right) B_{1_o}$$

$$b_2 \equiv -\left(\frac{\eta}{\Omega}\sin\frac{\eta\pi}{\Omega}\right) B_{2_o}$$

$$b_3 \equiv -\left(\frac{\delta}{\Omega}e^{-\frac{\delta\pi}{\Omega}}\right)\left[\left(\sin\frac{\eta\pi}{\Omega}\right) B_{1_o} + \left(\cos\frac{\eta\pi}{\Omega}\right) B_{2_o}\right] .$$

Since $C_o = x(\frac{\pi}{\Omega}) = -x_o$, Eq. (3.23) reduces to

$$\Delta x'_o = -C_1\Delta x_o - C_2\Delta\dot{x}_o - C_3\Delta T - C_4\Delta\tau_o . \tag{3.24}$$

The time required by the particle to travel from one side of the container to the other side equals the absolute distance traveled, divided by the absolute velocity. Hence

$$\frac{(\pi + \Delta t'_o) - \Delta t_o}{\Omega} = \frac{\left[y(\pi + \Delta t'_o) + x(\pi + \Delta t'_o)\right] - \left[y(\Delta t_o) + x(\Delta t_o)\right]}{-(v_o + \Delta v_o)} . \tag{3.25}$$

Substituting the values of the quantities on the left hand side of (3.25), then

$$\Delta T = \frac{\Omega}{v_o}\Delta x'_o + \frac{\Omega}{v_o}\Delta x_o - \frac{\pi}{v_o}\Delta v_o . \tag{3.26}$$

Replacing ΔT in (3.24) by its value from (3.26), we obtain

$$\Delta x'_o = d_5\,\Delta x_o + \frac{v_o}{\Omega} d_2\,\Delta\dot{x}_o - C_3 d_3\,\Delta v_o + \frac{v_o}{\Omega} d_4 \Delta\tau_o \tag{3.27}$$

where

$$d_5 \equiv -\left(\frac{\Omega C_3 + C_1 v_o}{v_o + \Omega C_3}\right)$$

$$d_2 \equiv -\frac{\Omega C_2}{v_o + \Omega C_3}$$

$$d_3 \equiv -\frac{\pi}{v_o + \Omega C_3}$$

$$d_4 \equiv -\frac{\Omega C_4}{v_o + \Omega C_3} \quad .$$

If (3.27) is now substituted in (3.26), then

$$\Delta T = d_1 \Delta x_o + d_2 \Delta \dot{x}_o + d_3 \Delta v_o + d_4 \Delta \tau_o \tag{3.28}$$

where $d_1 \equiv \dfrac{\Omega(1-C_1)}{v_o + \Omega C_3}$.

Since $\Delta\tau'_o = \Delta\tau_o + \Delta T$

then by using (3.28)

$$\Delta\tau'_o = d_1 \Delta x_o + d_2 \Delta \dot{x}_o + d_3 \Delta v_o + (1 + d_4)\Delta\tau_o \ . \tag{3.29}$$

Using Eq. (3.14) to find the velocity at $t = \left(\frac{\pi + \Delta t'_o}{\Omega}\right)_-$, then

$$\begin{aligned} \dot{x}\left(\frac{\pi + \Delta t'_o}{\Omega}\right)_- = e^{-\delta\left(\frac{\pi+\Delta T}{\Omega}\right)} & \left[-\delta \sin \eta\left(\frac{\pi + \Delta T}{\Omega}\right) + \eta \cos \eta\left(\frac{\pi + \Delta T}{\Omega}\right)\right] B_1(0 + \Delta t_o) \\ & + \left[-\delta \cos \eta\left(\frac{\pi + \Delta T}{\Omega}\right) - \eta \sin \eta\left(\frac{\pi + \Delta T}{\Omega}\right)\right] B_2(0 + \Delta t_o) \\ & + A\Omega \cos(\pi + \Delta T + \tau_o + \Delta\tau_o) \end{aligned}$$

$$= \rho_o + \rho_1 \Delta x_o + \rho_2 \Delta \dot{x}_o + \rho_3 \Delta T + \rho_4 \Delta\tau_o \tag{3.30}$$

where

$$\rho_o \equiv g_o e^{-\frac{\delta\pi}{\Omega}} - A\Omega \cos \tau$$

$$\rho_1 \equiv g_1 e^{-\frac{\delta\pi}{\Omega}}$$

$$\rho_2 = g_2 e^{-\frac{\delta\pi}{\Omega}}$$

$$\rho_3 \equiv (g_3 - \frac{\delta}{\Omega} g_o) e^{-\frac{\delta\pi}{\Omega}} + A\Omega \sin \tau_o$$

$$\rho_4 \equiv g_4 e^{-\frac{\delta\pi}{\Omega}} + A\Omega \sin \tau_o$$

$$g_o \equiv -(f_1 B_{1_o} + f_2 B_{2_o})$$

$$g_1 \equiv -(f_1 \frac{\delta}{\eta} + f_2)$$

$$g_2 \equiv -\frac{f_1}{\eta}$$

$$g_3 \equiv \frac{\eta}{\Omega}(f_1 B_{2_o} - f_2 B_{1_o})$$

$$g_4 \equiv f_2 A \cos \tau - a f_1$$

$$f_1 \equiv \delta \sin \frac{\eta\pi}{\Omega} - \eta \cos \frac{\eta\pi}{\Omega}$$

$$f_2 \equiv \delta \cos \frac{\eta\pi}{\Omega} + \eta \sin \frac{\eta\pi}{\Omega} .$$

From Eqs. (2.7) and (2.8),

$$\dot{x}_+ = k_1 \dot{x}_- + k_2 v_-$$

and

$$v_+ = k_3\,\dot{x}_- + k_4\,v_-$$

where

$$k_1 \equiv \left(\frac{1-\mu e}{1+\mu}\right)$$

$$k_2 \equiv \frac{\mu(1+e)}{1+\mu}$$

$$k_3 \equiv \left(\frac{1+e}{1+\mu}\right)$$

$$k_4 \equiv \left(\frac{\mu-e}{1+\mu}\right)$$

Hence

$$\dot{x}\left(\frac{\pi+\Delta t'_o}{\Omega}\right)_+ = k_1\left[\rho_1\Delta x_o + \rho_2\Delta\dot{x}_o + \rho_3\Delta T + \rho_4\Delta\tau_o\right] - k_2\Delta v_o$$
$$+ k_1\rho_o - k_2 v_o \quad ,$$

and

$$v\left(\frac{\pi+\Delta t'_o}{\Omega}\right)_+ = k_3\left[\rho_1\Delta x_o + \rho_2\Delta\dot{x}_o + \rho_3\Delta T + \rho_4\Delta\tau_o\right]$$
$$- k_4\Delta v_o + (k_3\rho_o - k_4 v_o) \quad .$$

Noting that

$$k_1\rho_o - k_2 v_o = k_1\dot{x}\left(\frac{\pi}{\Omega}\right)_- + k_2(-v_o) = -\dot{x}_o$$

and

$$k_3\rho_o - k_4 v_o = k_3\dot{x}\left(\frac{\pi}{\Omega}\right)_- + k_4 v\left(\frac{\pi}{\Omega}\right)_- = v_o$$

and replacing ΔT by its value from (3.28), then

$$\Delta\dot{x}'_o = -k_1(\rho_1+\rho_3 d_1)\Delta x_o - k_1(\rho_2+\rho_3 d_2)\Delta\dot{x}_o$$
$$+ (k_2 - k_1\rho_3 d_3)\Delta v_o - k_1(\rho_4+\rho_3 d_4)\Delta\tau_o \quad , \tag{3.31}$$

and

$$\Delta v_o' = k_3(\rho_1 + \rho_3 d_1)\Delta x_o + k_3(\rho_2 + \rho_3 d_2)\Delta \dot{x}_o$$

$$+ (k_3 \rho_3 d_3 - k_4)\Delta v_o + k_3(\rho_4 + \rho_3 d_4)\Delta \tau_o \quad . \tag{3.32}$$

Equations (3.27), (3.31), (3.32), and (3.29) can be expressed in matrix form as:

$$\begin{Bmatrix} \Delta x_o' \\ \Delta \dot{x}_o' \\ \Delta v_o' \\ \Delta \tau_o' \end{Bmatrix} = \begin{bmatrix} d_5 & \frac{v_o}{\Omega} d_2 & -c_3 d_3 & \frac{v_o}{\Omega} d_4 \\ -k_1(\rho_1+\rho_3 d_1) & -k_1(\rho_2+\rho_3 d_2) & k_2-k_1\rho_3 d_3 & -k_1(\rho_4+\rho_3 d_4) \\ k_3(\rho_1+\rho_3 d_1) & k_3(\rho_2+\rho_3 d_2) & k_3\rho_3 d_3-k_4 & k_3(\rho_4+\rho_3 d_4) \\ d_1 & d_2 & d_3 & 1+d_4 \end{bmatrix} \begin{Bmatrix} \Delta x_o \\ \Delta \dot{x}_o \\ \Delta v_o \\ \Delta \tau_o \end{Bmatrix} \cdot \tag{3.33}$$

Special case ($\Omega = 1, \ \delta = 0$)

Using Eq. (2.34) and noting that $\tau_o = -\frac{\pi}{2}$, then

$$x = (x_o + \Delta x_o)\cos t + (\dot{x}_o + \Delta \dot{x}_o)\sin t - \frac{t}{2}\sin(t + \Delta\tau_o) \quad , \tag{3.34}$$

$$\dot{x} = -(x_o + \Delta x_o)\sin t + (\dot{x}_o + \Delta \dot{x}_o)\cos t - \frac{1}{2}\sin(t + \Delta\tau_o)$$

$$- \frac{t}{2}\cos(t + \Delta\tau_o) \quad , \tag{3.35}$$

$$(0 + \Delta t_o)_+ \le t \le (\pi + \Delta t_o')_- \quad .$$

Hence

$$x(\pi + \Delta t_o')_- = (x_o + \Delta x_o)\cos(\pi + \Delta T) + (\dot{x}_o + \Delta \dot{x}_o)\sin(\pi + \Delta T)$$

$$- \frac{1}{2}(\pi + \Delta T)\sin(\pi + \Delta T + \Delta\tau_o) \quad ,$$

which results in

$$\Delta x_o' = \Delta x_o + (\dot{x}_o - \frac{\pi}{2})\Delta T - \frac{\pi}{2}\Delta\tau_o \quad .$$

Since

$$\Delta T = \frac{1}{v_o}\Delta x_o' + \frac{1}{v_o}\Delta x_o - \frac{\pi}{v_o}\Delta v_o$$

$$= \frac{2\Delta x_o - \pi\Delta v_o - \frac{\pi}{2}\Delta\tau_o}{\frac{2(1+\mu)}{1+e}v_o} \qquad (3.36)$$

then

$$\Delta x_o' = -k_4\Delta x_o + \frac{\pi}{2}(k_1+k_2+k_4)\Delta v_o - \frac{\pi}{4}k_3\Delta\tau_o \ , \qquad (3.37)$$

and

$$\Delta\tau_o' = \Delta\tau_o + \Delta T = \frac{4}{\pi}k_2\Delta x_o - 2k_2\Delta v_o + k_1\Delta\tau_o \ . \qquad (3.38)$$

Now since

$$\dot{x}(\pi+\Delta t_o')_+ = k_1\left[(x_o+1)\Delta T - \Delta\dot{x}_o + \frac{\Delta\tau_o}{2}\right] - k_2\Delta v_o$$

$$- k_1(\dot{x}_o - \frac{\pi}{2}) - k_2 v_o \qquad (3.39)$$

and

$$v(\pi+\Delta t_o')_+ = k_3\left[(x_o+1)\Delta T - \Delta\dot{x}_o + \frac{\Delta\tau_o}{2}\right] - k_4\Delta v_o$$

$$- k_3(\dot{x}_o - \frac{\pi}{2}) - k_4 v_o \ , \qquad (3.40)$$

by noting that

$$- k_1(\dot{x}_o - \frac{\pi}{2}) - k_2 v_o = -\dot{x}_o$$

and

$$- k_3(\dot{x}_o - \frac{\pi}{2}) - k_4 v_o = v_o \ ,$$

and using the value of ΔT in Eq. (3.36), Eqs. (3.39) and (3.40) reduce to

$$\Delta\dot{x}_o' = -\frac{4}{\pi} k_1 k_2 q \Delta x_o + k_1 \Delta\dot{x}_o$$
$$+ k_2\left[1 + 2k_1 q\right]\Delta v_o - k_1(\frac{1}{2} - k_2 q)\Delta\tau_o \tag{3.41}$$

and

$$\Delta v_o' = \frac{4}{\pi} k_2 k_3 q \Delta x_o - k_3 \Delta\dot{x}_o$$
$$- \left[k_4 + 2k_2 k_3 q\right]\Delta v_o + k_3(\frac{1}{2} - k_2 q)\Delta\tau_o \quad , \tag{3.42}$$

where

$$q \equiv (1 + \frac{\pi^2}{8\mu} - \frac{d}{2}) = (x_o + 1) \quad .$$

Equations (3.37), (3.41, (3.42) and (3.38) can be expressed in matrix form as:

$$\begin{Bmatrix} \Delta x_o' \\ \Delta\dot{x}_o' \\ \Delta v_o' \\ \Delta\tau_o' \end{Bmatrix} = \begin{bmatrix} -k_4 & 0 & \frac{\pi}{2}(k_1+k_2+k_4) & -\frac{\pi}{4}k_3 \\ -\frac{4}{\pi}k_1k_2q & k_1 & k_2(1+2k_1q) & -k_1(\frac{1}{2} - k_2q) \\ \frac{4}{\pi}k_2k_3q & -k_3 & -(k_4+2k_2k_3q) & k_3(\frac{1}{2} - k_2q) \\ \frac{4}{\pi}k_2 & 0 & -2k_2 & k_1 \end{bmatrix} \begin{Bmatrix} \Delta x_o \\ \Delta\dot{x}_o \\ \Delta v_o \\ \Delta\tau_o \end{Bmatrix} \cdot \tag{3.43}$$

3.3. Stability Boundaries

The stability boundaries are the curves on which the modulus of the largest eigenvalue(s) equals unity.

The characteristic polynomial of the matrix P is

$$\begin{vmatrix} P_{11}-\lambda & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22}-\lambda & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33}-\lambda & P_{34} \\ P_{41} & P_{42} & P_{43} & P_{44}-\lambda \end{vmatrix} = 0 \tag{3.44}$$

which can be put in the form

$$\varphi(\lambda) = \lambda^4 - \alpha_1\lambda^3 + \alpha_2\lambda^2 - \alpha_3\lambda + \alpha_4 = 0 . \tag{3.45}$$

From the theory of matrices, it is known that[19] if the eigenvalues of P are λ_1, λ_2, λ_3, λ_4, then:

$$\left.\begin{aligned}
\alpha_1 &= \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = \sum_{i=1}^{4} P_{ii} = \text{Trace of P} \\
\alpha_2 &= \lambda_1(\lambda_2 + \lambda_3 + \lambda_4) + \lambda_2(\lambda_3 + \lambda_4) + \lambda_3\lambda_4 \\
\alpha_3 &= \lambda_1(\lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4) + \lambda_2\lambda_3\lambda_4 \\
\alpha_4 &= \lambda_1\lambda_2\lambda_3\lambda_4 = \text{determinant of P} .
\end{aligned}\right\} \tag{3.46}$$

Guided by some knowledge about the behavior of the eigenvalues of P, we will assume that on the boundary, one of two cases occurs:

a) $|\lambda_1| = 1$

b) $|\lambda_1| = |\lambda_2| = 1 \;;\; \lambda_1 = \overline{\lambda}_2$ (complex conjugates)

Case a):

If one of the eigenvalues (e.g., λ_1) is real and equal to ± 1, then from (3.45) and (3.46)

$$\frac{1 + \alpha_2 + \alpha_4}{\alpha_1 + \alpha_3} = \lambda_1 = \pm 1 . \tag{3.47}$$

Case b):

$$\lambda_1 = a_o + ib_o , \quad \lambda_2 = a_o - ib_o \;;\; a_o^2 + b_o^2 = 1 .$$

By using (3.45) and (3.46), it is found that

$$(\alpha_2 - 1 - \alpha_4)(\alpha_4 - 1)^2 = (\alpha_3 - \alpha_1)(\alpha_1\alpha_4 - \alpha_3) . \tag{3.48}$$

Due to the nature of the expressions for the α's, Eqs. (3.47) and (3.48) are transcendental equations. Their solutions (obtained e.g. by iteration) will furnish the stability boundaries.

For the special case of $\Omega = 1$, $\delta = 0$, the expressions for the α's simplify considerably, and with the help of (3.43) they are found to be:

$$\left.\begin{aligned} \alpha_1 &= 2(k_1 - k_4) - 2k_2k_3q \ , \\ \alpha_2 &= \left[(k_1 - k_4)^2 + 2e + k_2k_3\right] - 4k_2k_3q \ , \\ \alpha_3 &= \left[2e(k_1 - k_4) + k_2k_3\right] - 2k_2k_3q \ , \\ \alpha_4 &= e^2 \ . \end{aligned}\right\} \qquad (3.49)$$

Considering first the case of $\lambda = \pm 1$, then from (3.47) and (3.44) it is found that

$$\left[(1+e)^2 + (k_1 - k_4)^2 + k_2k_3\right] - 4k_2k_3q = \lambda\left\{\left[2(1+e)(k_1 - k_4) + k_2k_3\right] - 4k_2k_3q\right\} \qquad (3.50)$$

which will involve the stability boundaries only if $\lambda = -1$. Thus, for $\lambda = -1$, (3.50) reduces to

$$q = q_a \equiv \frac{2+\mu}{4\mu} \ ,$$

consequently,

$$d = d_a \equiv \frac{6\mu + \pi^2 - 4}{4\mu} \approx 1.5\left(1 + \frac{1}{\mu}\right) \ . \qquad (3.51)$$

For the case $\lambda = \overline{\lambda}_2$, $|\lambda_1| = 1$, (3.48) and (3.44) result in

$$q = q_b \equiv \frac{N(q)}{D(q)} \qquad (e < 1) \qquad (3.52)$$

where

$$N(q) \equiv (1-e)^2 \left\{ \left(\frac{1-\mu}{1+\mu}\right)^2 \left[(1+e)^2 + 4e \right] - (1-e)^2 \right\}$$

$$+ \mu \left(\frac{1+e}{1+\mu}\right)^2 \left[2 \left(\frac{1-e}{1+e}\right) \left(\frac{1-\mu}{1+\mu}\right) - \frac{\mu}{(1+\mu)^2} + (1-e)^2 \right] ,$$

$$D(q) \equiv \frac{2\mu^2}{(1+\mu)^2} (1-e^2) \left[\left(\frac{1+e}{1+\mu}\right)^2 + 4 \frac{(1-e^2)}{1+\mu} \right] .$$

With q determined from (3.52), d can be found from

$$d = d_b \equiv 2(1 + \frac{\pi^2}{8\mu} - q_b) .$$

4. EXPERIMENTAL STUDIES

4.1. Introduction

The experiments that were performed during the course of this investigation were primarily motivated by:

a) lack of sufficient information concerning the general behavior of the system for a wide range of its parameters,

b) possible design problems that may be encountered in the actual construction of such a system,

c) need for a stability criterion for 2 impacts/cycle motion,

d) need to evaluate the efficiency of the system as a vibration damper,

e) desire to explore its effectiveness for random excitation.

In order to obtain some information relevant to these matters, the following experiments and studies were conducted as described below:

1 - experiments with a mechanical model,

2 - experiments with an electric analog,

3 - numerical studies involving a digital computer.

4.2. Experiments with a Mechanical Model

The main purpose for conducting this type of experiments was to gain some knowledge concerning:

a) qualitative behavior of the system,

b) possible design problems,

c) variation of the coefficient of restitution with time (i.e. number of impacts),

d) motion of system with other than steady two impacts per cycle.

A schematic diagram of the mechanical model that was used and a photograph of the actual structure are shown in Fig. 4. 1 and 4. 2, respectively.

Since the qualitative response of a single degree of freedom oscillator is not altered if the excitation is applied to the base instead of directly to the mass, the former type of excitation was used in this case, as a matter of convenience.

The mass M was primarily a rectangular box with rigid stops at its ends, that constrained the movement of the frictionless solid particle m to oscillate horizontally within a certain clearance. The particle was a hardened steel ball, of a type used in ball bearings, and the stops which it impinged upon were also made of hardened steel so as to obtain a high coefficient of restitution. The coefficient of restitution as used in the experiment was approximately 0. 8.

The relative motion between the base (which was excited by an electromagnetic shaker) and M was monitored by means of a linear variable differential transformer[20] whose output was recorded by a direct inking oscillograph.

Typical records of the relative motion obtained in this manner are shown in Fig. 4. 3.

Under steady state conditions with 2 impacts/cycle, the value of the coefficient of restitution e can be found with the help of Eqs. (2. 12) and (2. 13) to be

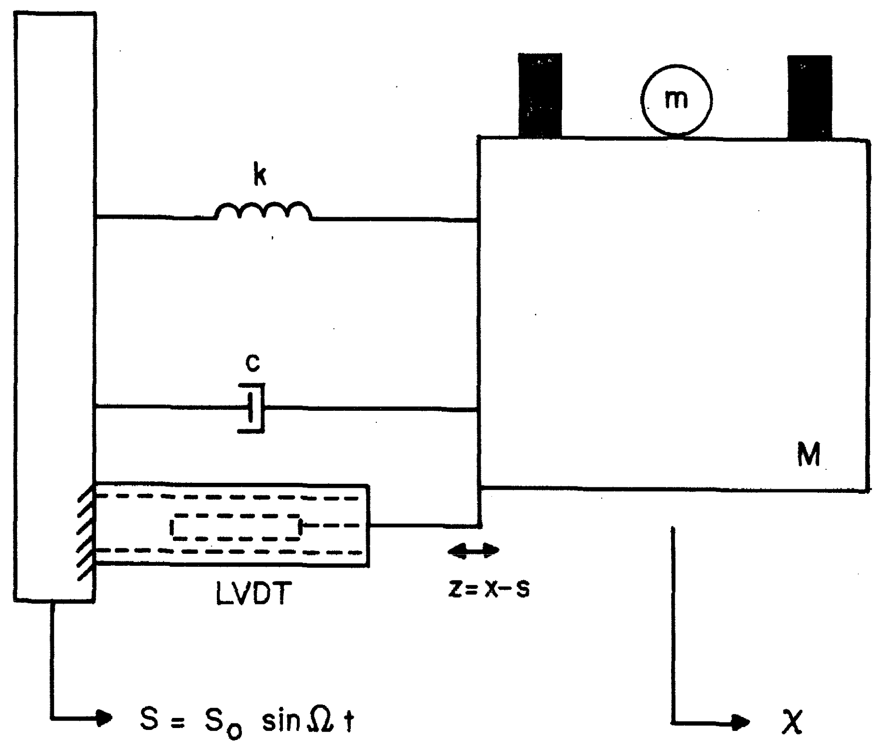
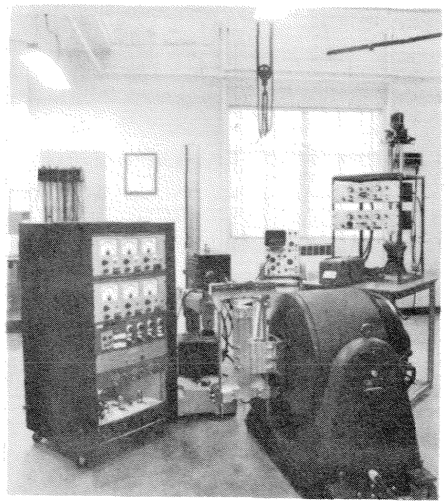
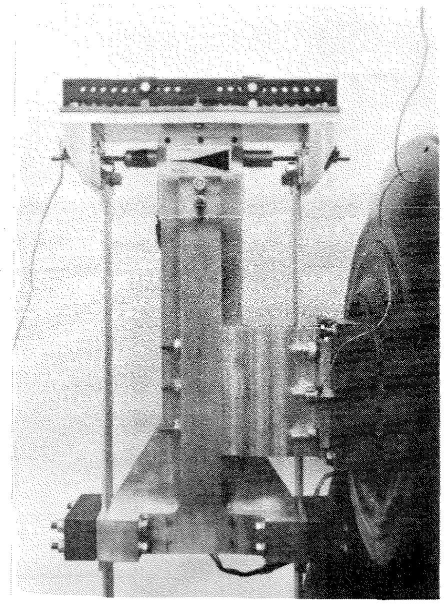


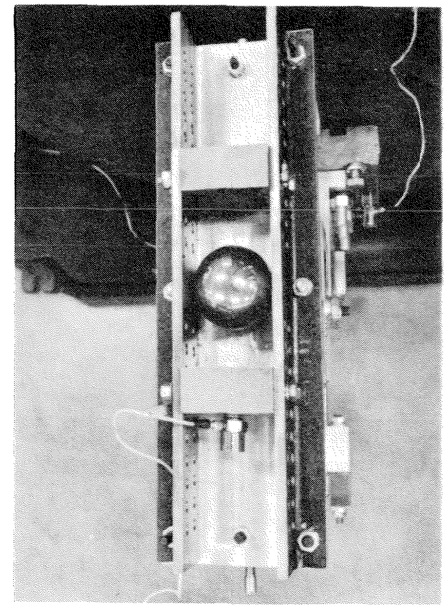
Fig. 4.1. Mechanical model.



General View

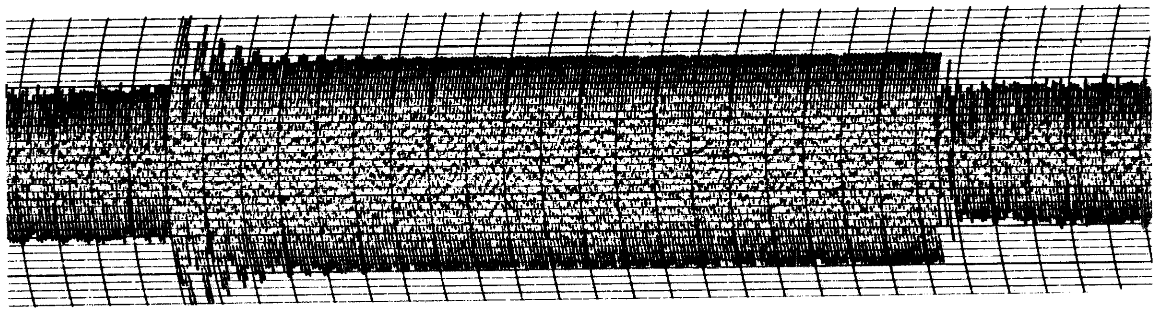


Side View

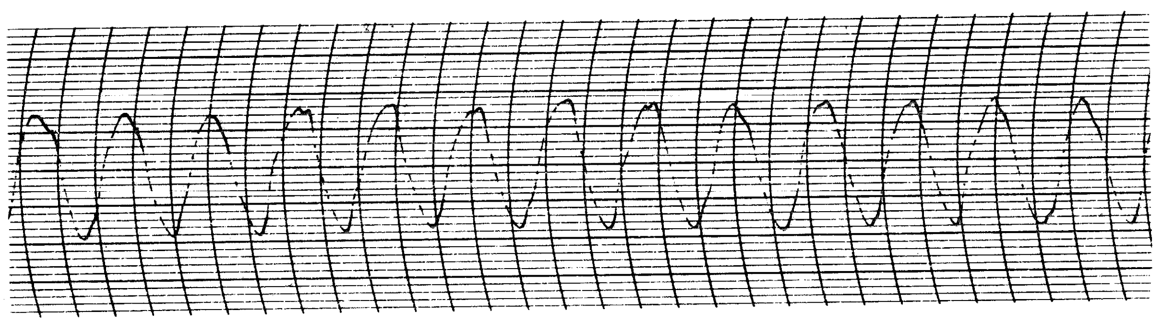


Top View

Fig. 4.2. Photograph of mechanical model.



(a)



(b)

Fig. 4.3. Relative motion of M: a) effects of removing m, b) approximately steady 2 impacts/cycle.

$$e = \frac{1 - (1+2\mu)\frac{\dot{x}_a}{\dot{x}_b}}{(1+2\mu) - \frac{\dot{x}_a}{\dot{x}_b}} , \qquad (4.1)$$

The velocity ratio in Eq. (4. 1) was obtained by integrating with respect to time the output of a piezoelectric accelerometer attached to M. The integration was accomplished by using an integrating network in conjunction with an operational amplifier.

As a result of this phase of the investigation, which was conducted in the Dynamics Laboratory of the California Institute of Technology, the following observations were made:

a) The excessive (as measured by the human ear) noise level in the vicinity of an operating system resulting from the impacts, especially when the colliding surfaces are hardened, is of such an intensity as to require muffling, if the damper is to function over an extended period.

b) Since plastic deformation can result if the relative velocity of the colliding surfaces is more than 1 ft/sec.[21, 22], this phenomena is pertinent to the problem at hand, where the impact velocity is relatively high. In situations where there is appreciable plastic deformation, continuous operation of the damper may cause e to become a function of time--an undesirable state if the stability of periodic motion is of concern and if the stability boundaries are dependent on e.

It was found that by using hardened steel for the colliding surfaces, there will be no significant change in the value of e.

c) Although horizontal operation of the damper is simpler from the design viewpoint, other positions can also be accommodated (e.g. the particle may be attached to the tip of a soft cantilever beam rigidly connected to M).

d) In some cases, even though the 2 impacts/cycle motion is not stable in the strict mathematical sense, the amplitude of the response is nearly constant and appreciably less than the resulting amplitude when the damper is removed, as shown in Fig. 4.3(a).

Figure 4.3(b) shows the effects of setting $\mu = 0$ (by removing the particle from its container) while M is vibrating. Obviously, the increased amplitude equals that attained by an equivalent single degree of freedom system subjected to the same excitation. As soon as μ is returned to its former value, the motion of M resumes its former state.

Except for the expected transients that occur when μ is changed, even such a large "perturbation" does not alter the characteristics of the motion, provided that (as it is in this case) the system is not operating in the proximity of the stability boundaries.

e) The actual wave form of the response is approximately sinusoidal, and the assumption that the velocity changes discontinuously is justifiable, as shown in Fig. 4.4 (x was measured by an optical displacement follower).

f) In general, when 2 impacts/cycle motion is not stable, the resulting motion is irregular (i.e., neither the amplitude nor the frequency of impacts is constant), and sometimes it exhibits the

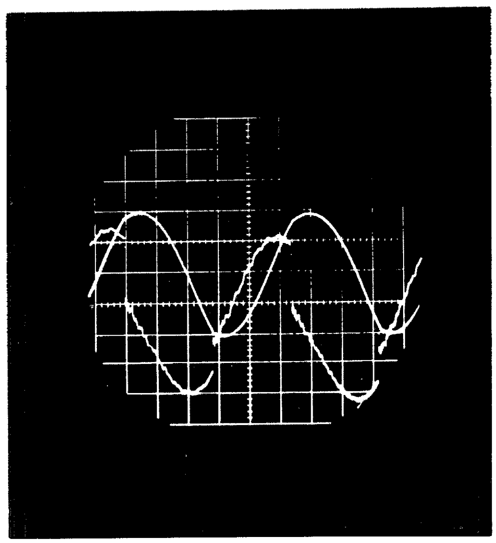


Fig. 4. 4. x and $\dot{x}$ as photographed from the screen of a cathode ray oscilloscope (x continuous, $\dot{x}$ discontinuous).

phenomenon of amplitude modulation (beat)[23].

4.3 Experiments with an Electric Analog

The main attractions of using an electric analog to study the behavior of this system were:

a) ease of varying the parameters of the model over a wide range,

b) simplicity of monitoring, measuring, and recording the variables of the system.

The objectives of this phase of the investigation were:

a) to find quantitative behavior of the system,

b) to study the general response of the system, and in particular stable periodic motions,

c) to evaluate the effects of small deviations from the mathematical model on the response.

Due to the limitations on the nature of the model, imposed by using real components in an electronic analog computer[24], some modifications had to be made in the electrical model. These modifications, while enhancing the faithfulness of the analog to the physical system, kept the behavior of the electrical model basically the same as the mathematical one.

The above mentioned modifications consisted of the following:

a) The infinitely rigid "mathematical" stops were replaced by very stiff springs, as compared to the main spring of M. This step is fully justified[25] since the response of a system to a symmetrical pulse is independent of the pulse shape as long as

$$\frac{\tau_p}{T} < \frac{1}{4}$$

where: τ_p = duration of pulse,

T = period of system.

b) The simplified concept of coefficient of restitution, i. e., relating the discontinuous values of the velocities immediately preceding and succeeding an impact, was replaced by a continuous process with the same end results.

A schematic diagram of the model whose behavior was simulated by means of an electronic analog computer, in the Applied Mechanics Department of the California Institute of Technology, is shown in Fig. 4. 5, and its equation of motion is

$$\left.\begin{aligned} M\ddot{x} &= -kx - c\dot{x} + F_o \sin \Omega t + k_3 g(y) + c_3 h(y, \dot{y}) \\ m\ddot{y} &= -m\ddot{x} \qquad\qquad\qquad\qquad - k_3 g(y) - c_3 h(y, \dot{y}) \end{aligned}\right\} \quad (4.2)$$

where $g(y)$ and $h(y, \dot{y})$ are nonlinear functions shown in Fig. 4. 6.

By a proper choice of k_3, the nonlinear springs can simulate a rigid barrier to any desired degree of accuracy. c_3 in conjunction with $h(y, \dot{y})$ provides means for simulating inelastic impacts, ranging from the completely plastic up to the elastic ones.

The circuits that were used to generate the nonlinear functions $g(y)$[26] and $h(y, \dot{y})$ are shown in Fig. 4. 7 and 4. 8, and their actual input-output characteristics are shown in Fig. 4. 9.

Equation (4. 2) can be put in the form

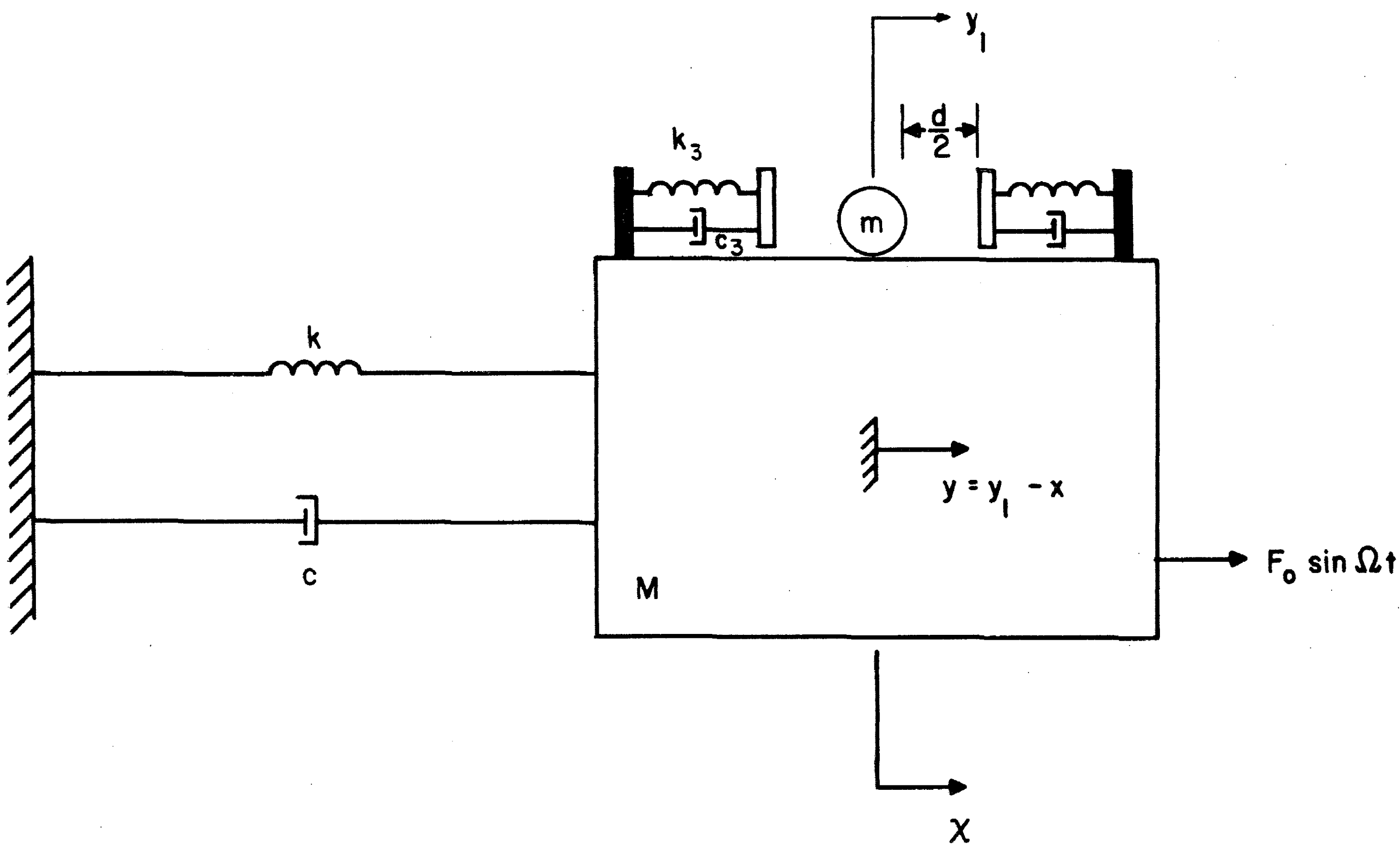


Fig. 4.5. Model studied by means of analog computer.

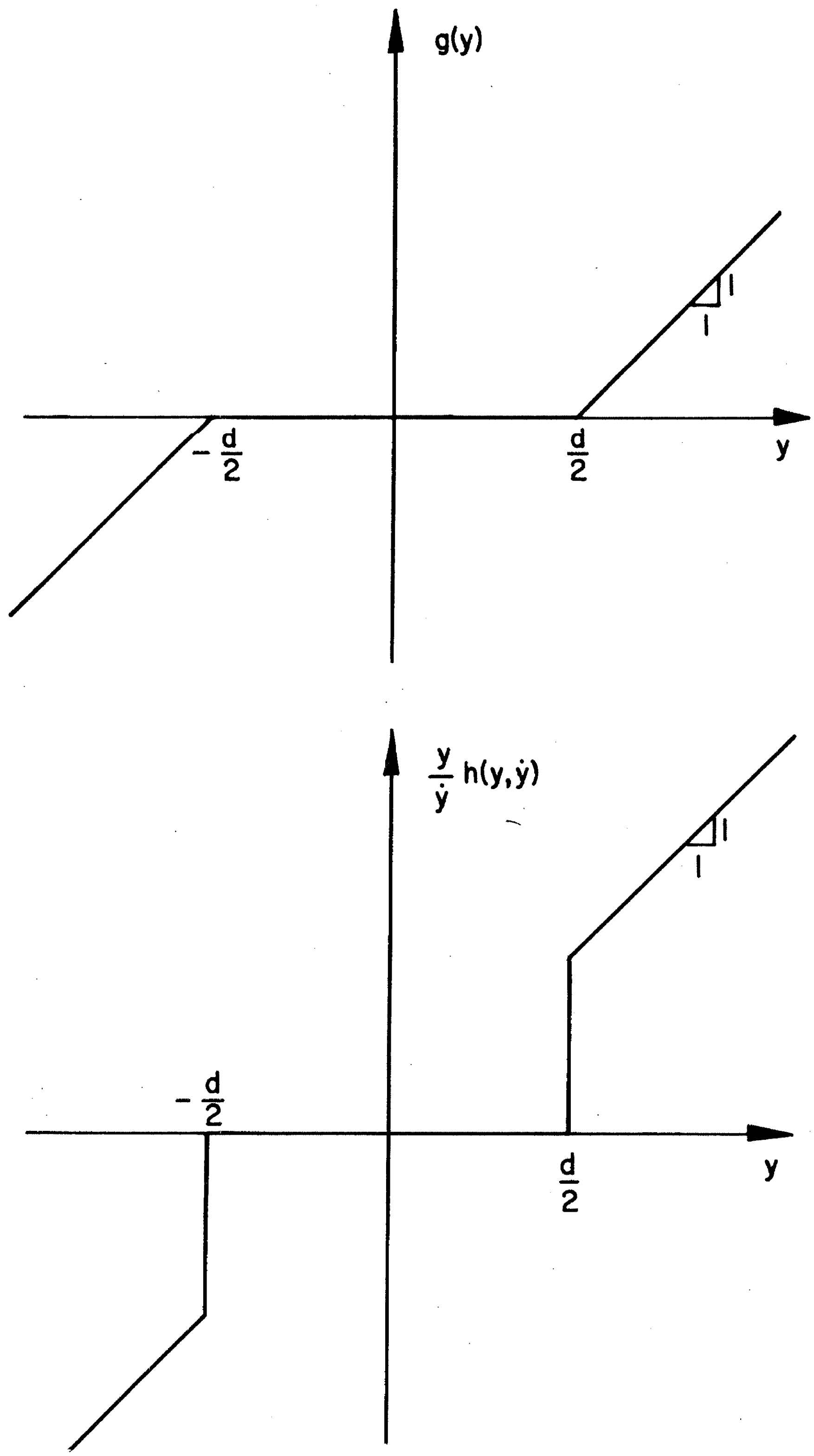


Fig. 4.6. Nonlinear functions.

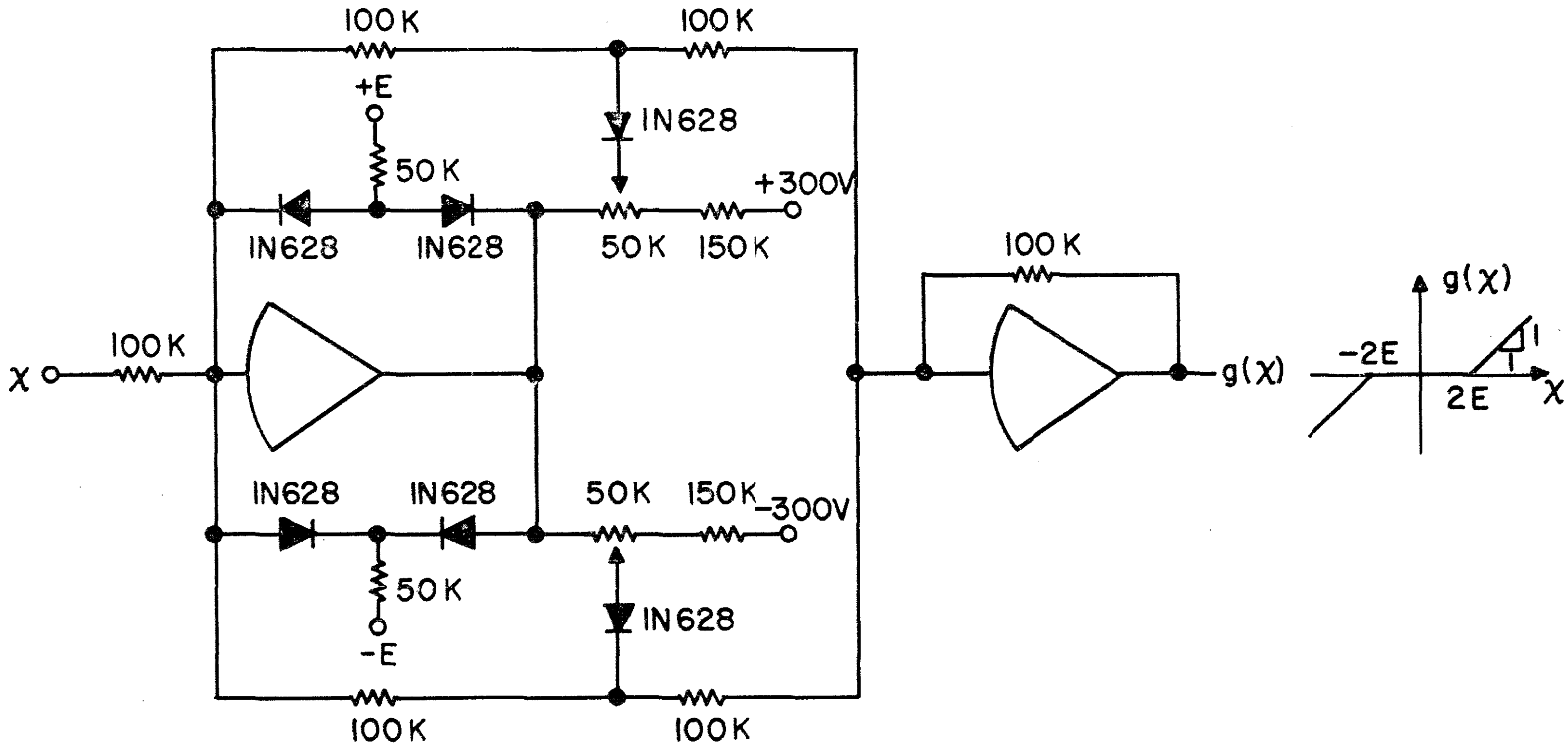


Fig. 4.7. Circuit for generating g(x).

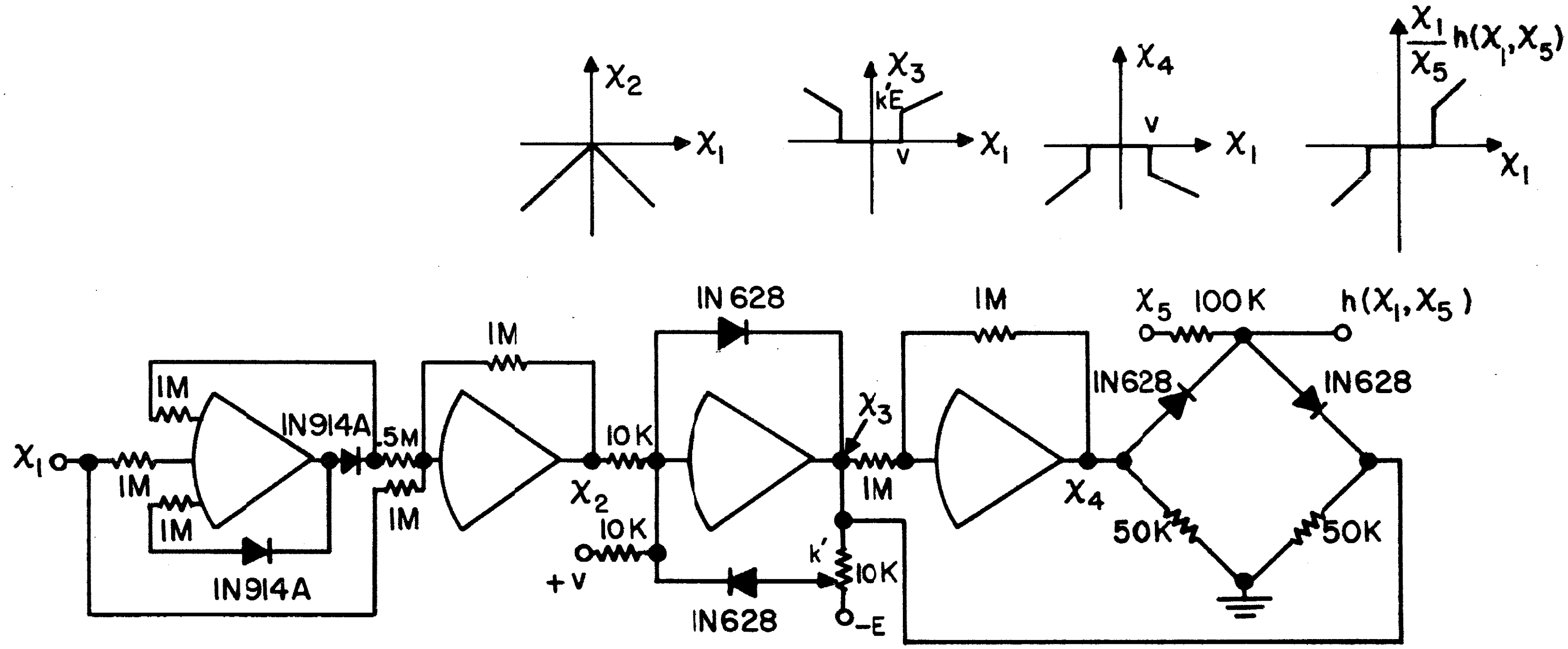


Fig. 4.8. Circuit for generating h(y, ẏ).

(a) (b)

Fig. 4. 9. Nonlinear functions as photographed from the screen of a cathode ray oscilloscope: (a) $g(y)$; (b) $\frac{y}{\dot{y}} h(y, \dot{y})$.

$$\left.\begin{aligned} \ddot{x} &= -\omega^2 x - 2\delta\omega\dot{x} + \frac{F_o}{M}\sin\Omega t + \mu\left[\omega_3^2 g(y) + 2\delta_3\omega_3 h(y,\dot{y})\right] \\ \ddot{y} &= -\ddot{x} \qquad\qquad\qquad\qquad\qquad\qquad - \left[\omega_3^2 g(y) + 2\delta_3\omega_3 h(y,\dot{y})\right] \end{aligned}\right\} \quad (4.3)$$

where: $\omega_3 = \sqrt{\dfrac{k_3}{m}}$,

$$\delta_3 = \frac{c_3}{2\sqrt{k_3 m}} \quad .$$

The computer diagram for Eq. (4.3) that was used in this investigation is shown in Fig. 4.10 (the notations used follow the usual ones[27]) and typical oscillograph records for the output of some of the amplifiers of Fig. 4.10 are shown in Fig. 4.11.

Each of the three pairs of records in Fig. 4.11 shows the simultaneous values of the labeled variables for the case (a typical one) where

$\omega = 10$ rad/sec	$r = 1$	$\mu = 0.10$
$\delta = 0.10$	$e = 0.80$	$\dfrac{d}{F_o/k} = 2$.

In Fig. 4.11(a), although the rapid velocity change is not as severe as that experienced by the actual structure (see Fig. 4.4), it still lies within the realm of impulsive motion for the reasons cited above.

Since e is defined as $e = -\dfrac{\dot{y}_+}{\dot{y}_-}$, its value in this particular case can be verified from the wave form of Fig. 4.11(b). Also from this figure, one can see that the wave forms of the relative velocity and displacement are not piece-wise linear--unlike $\dot{y}_1$ and y_1 whose wave forms in 2 impacts/cycle motion would be rectangular and triangular, respectively.

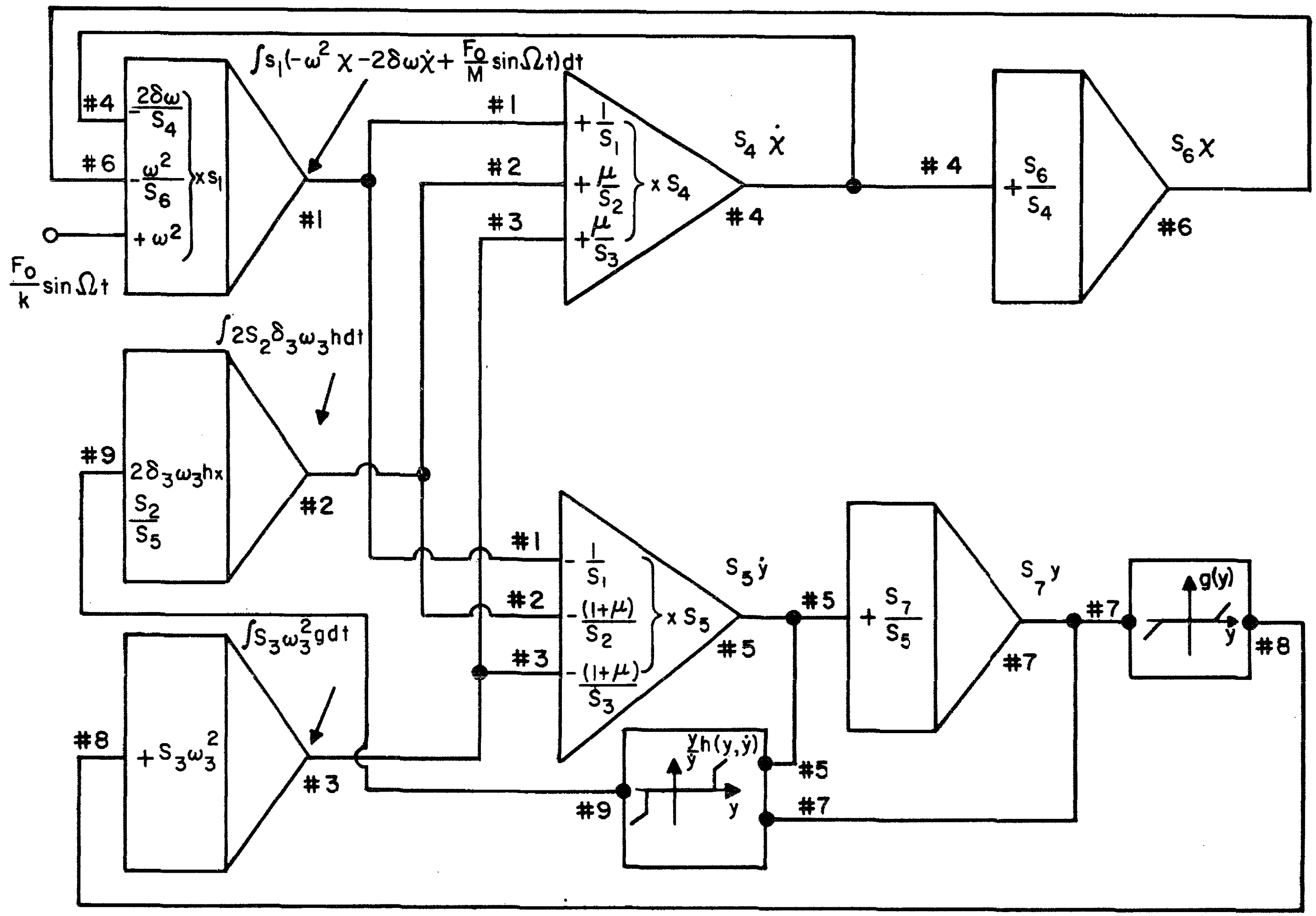


Fig. 4.10. Computer diagram for Eq. (4.3).

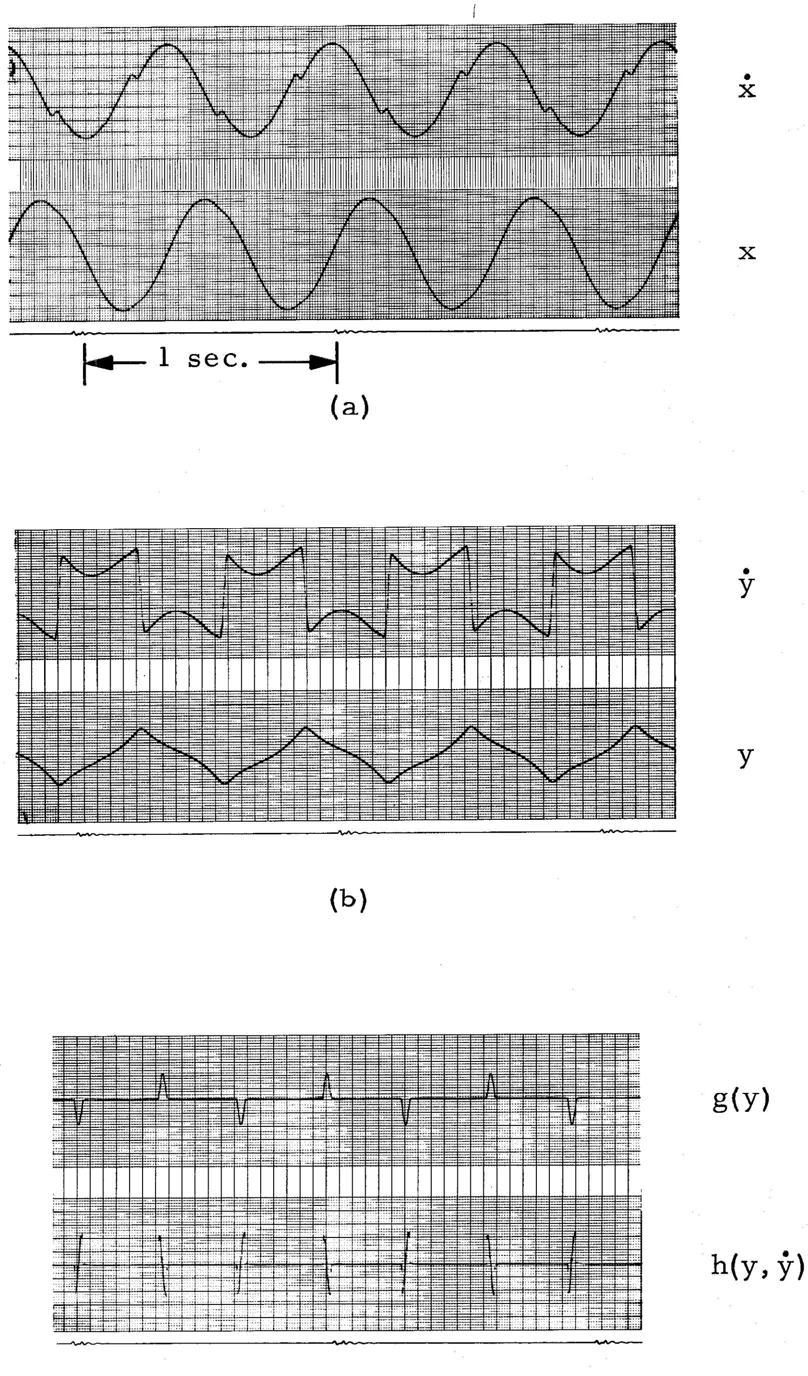


Fig. 4.11. Samples of recordings.

The outputs of the two nonlinear function generators are displayed in Fig. 4.11(c). It is obvious from the output of g(y) that the impulse resulting from the collision of the particle with the spring k_3 is symmetric and that $\frac{\tau_p}{T} < \frac{1}{20}$. The simultaneous output of g(y) and $h(y,\dot{y})$ shows that the actual performance of the dashpot c_3 is as intended, since it is in operation only while m is in contact with k_3.

The analog computer simulation of the removal of the particle (setting $\mu = 0$) while M is in motion is shown in Fig. 4.12 which, basically, duplicates the response of the actual structure.

The limit cycle for 2 impacts/cycle motion of an operating system is exhibited in Fig. 4.13(a) and compared to the phase diagram of a single degree of freedom oscillator (i.e. same system but with $\mu = 0$) in Fig. 4.13(b).

In Section 4.5 the close agreement between experimental results and the ones predicted by the theory developed in Chapter 2 is demonstrated for a particular set of the parameters.

The analog computer was also used to study the effects of small amount of friction and a weak linear spring between m and M (Fig. 4.14). As expected, it was found that small values of c_2 and k_2 have no appreciable influence on the behavior of the system.

That the impact damper is also effective in reducing the vibrations resulting from random forcing functions was found by simulating the behavior of the system on the analog computer. Fig. 4.15 shows the comparative motion of a randomly forced oscillator without and with the damping device.

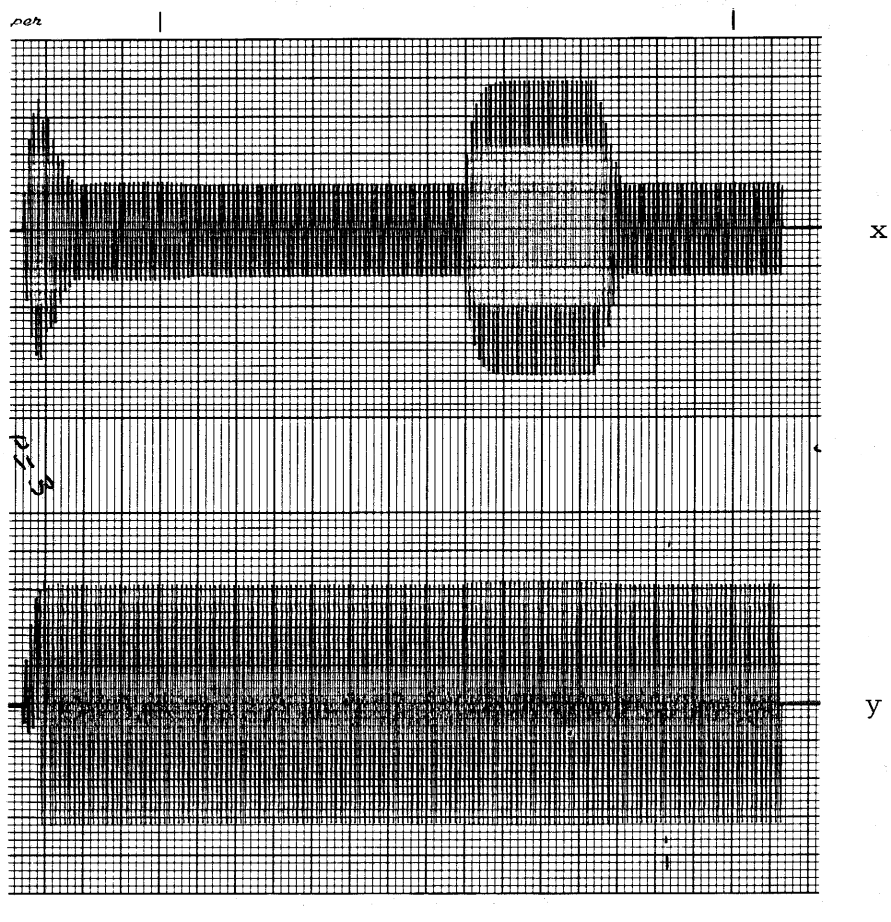


Fig. 4.12. Analog computer simulation of the removal of the particle.

(a) (b)

Fig. 4.13. Limit cycles of impact damper: (a) upper trace $x, \dot{x}$, lower trace $y, \dot{y}$; (b) $x, \dot{x}$ with $\mu = 0$ and $\mu \neq 0$.

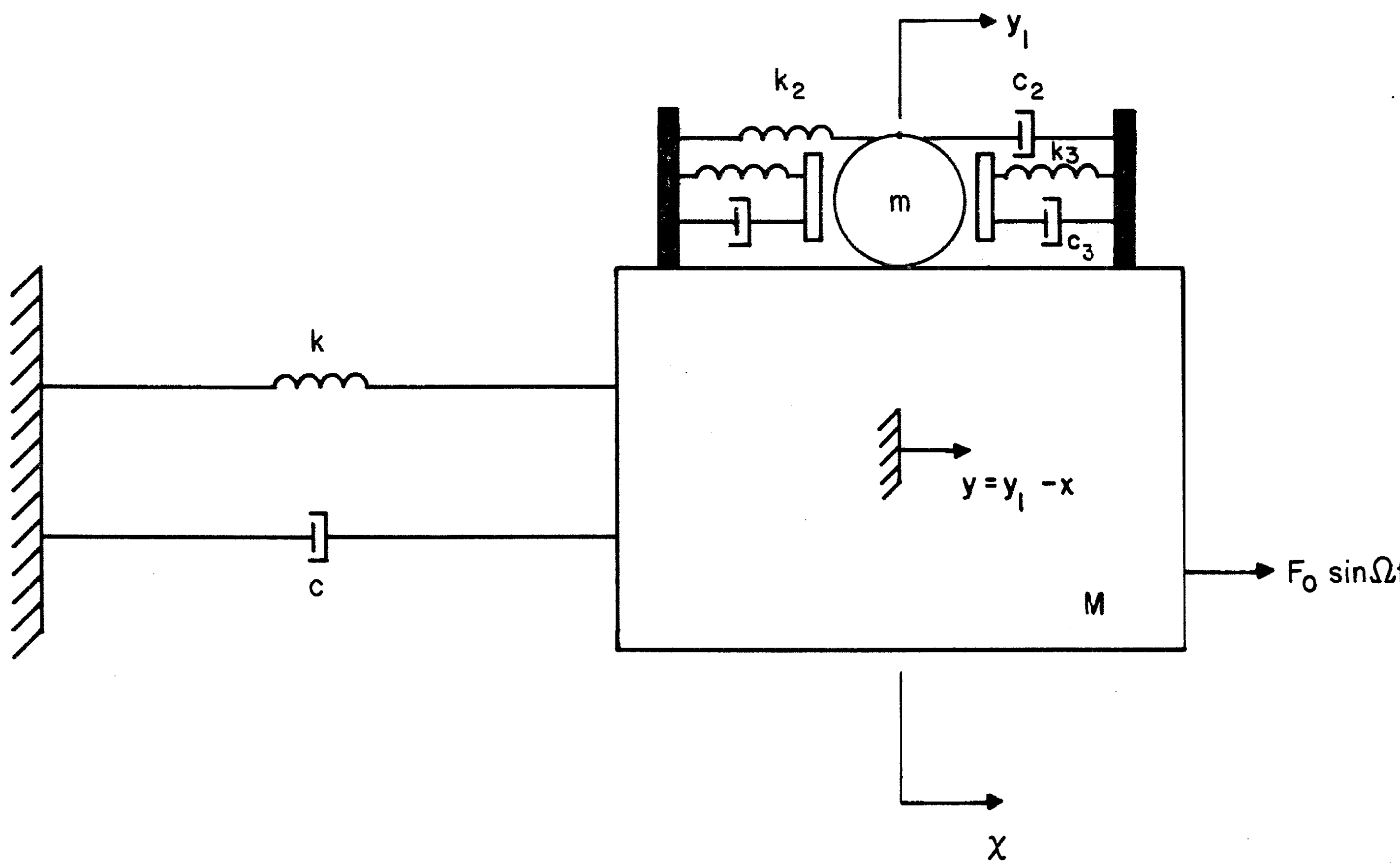
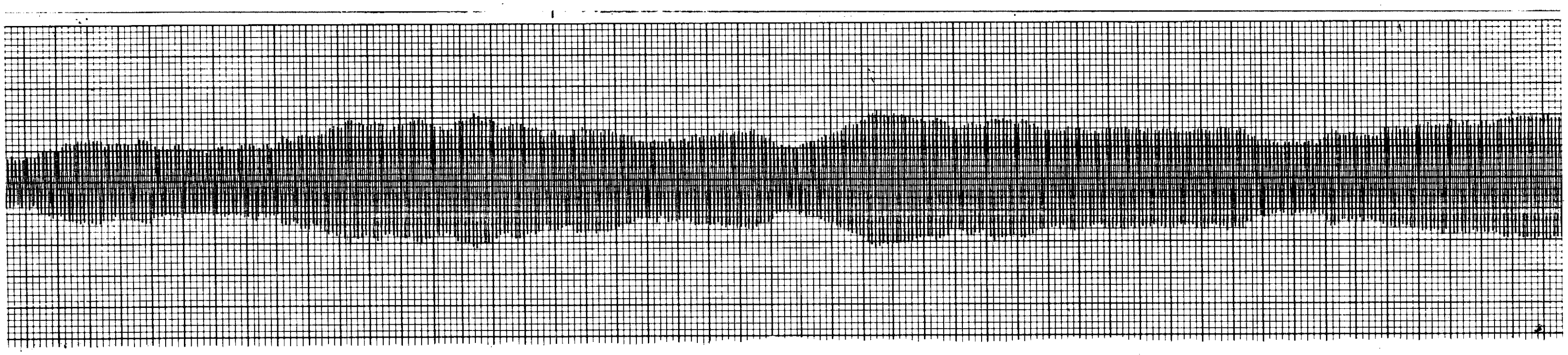
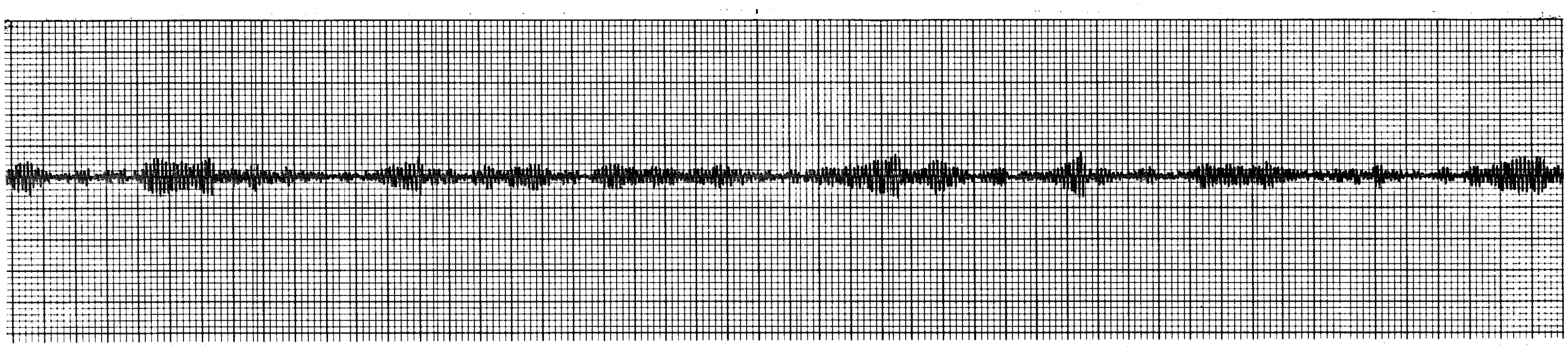


Fig. 4. 14. Model with friction and restoring force between M and m.



(a)



(b)

Fig. 4.15. Response of system to random excitation (ω = 10/rad/sec., δ = 0, e = 1); (a) μ = 0; (b) μ = 0.05.

4.4 Digital Computer Studies

An alternative approach to the above analytical studies is the direct step-by-step solution of the basic equation of motion on a digital computer. The disadvantages of this digital computer approach are the difficulty of exhibiting the results in a general form and certain computational problems in the investigation of marginal stability regions. The advantage is the fact that at least for specific cases a complete picture of system response can be obtained to any desired degree of accuracy.

The equation of motion of the mathematical model (Fig. 2.1), between impacts, is

$$\left.\begin{aligned} M\ddot{x} + c\dot{x} + kx &= F_o \sin \Omega t \\ \ddot{y} &= -\ddot{x} \; . \end{aligned}\right\} \qquad (4.4)$$

If immediately after the i^{th} impact at $t = t_i$

$$x(t_{i_+}) = x_i \; ; \quad y(t_{i_+}) = y_i \; ; \quad \dot{x}(t_{i_+}) = \dot{x}_i \; ; \quad \dot{y}(t_{i_+}) = \dot{y}_i$$

then the motion of M and m is described during the time interval from t_{i_+} to the time immediately preceding the next impact, $t_{(i+1)_-}$ by

$$\left.\begin{aligned} x &= e^{-\delta\omega(t-t_i)}\left[D_i \sin \eta\omega(t-t_i) + E_i \cos \eta\omega(t-t_i)\right] + A \sin(\Omega t - \psi) \\ y &= -x + (x_i + y_i) + (\dot{x}_i + \dot{y}_i)(t - t_i) \\ & \qquad\qquad t_{i_+} \le t \le t_{(i+1)_-} \end{aligned}\right\} \qquad (4.5)$$

where:

$$E_i = x_i - A \sin(\Omega t_i - \psi)$$

$$D_i = \frac{1}{\eta}\left[\delta E_i + \frac{\dot{x}_i}{\omega} - Ar \cos(\Omega t_i - \psi)\right] .$$

From the impact conditions at $t_{(i+1)_+}$,

$$
\begin{aligned}
x\left(t_{(i+1)_+}\right) &= x\left(t_{(i+1)_-}\right) \\
y\left(t_{(i+1)_+}\right) &= y\left(t_{(i+1)_-}\right) \quad ; \quad |y| = \frac{d}{2} \\
\dot{x}\left(t_{(i+1)_+}\right) &= \dot{x}\left(t_{(i+1)_-}\right) + k_2 \dot{y}\left(t_{(i+1)_-}\right) \\
\dot{y}\left(t_{(i+1)_+}\right) &= -e\,\dot{y}\left(t_{(i+1)_-}\right) \quad .
\end{aligned}
\tag{4.6}
$$

Conditions (4. 6) can now be used as new initial conditions in Eq. (4. 5) for the time interval $t_{(i+1)_+}$ to $t_{(i+2)_-}$. This process can be repeated over and over again so as to obtain the time behavior of the model.

A digital computer program to find the "exact" sequence of initial conditions and the resulting motion according to (4. 5), for any given set of parameters and "initial" initial conditions, was written in FORTRAN IV language, and executed by means of an IBM 7090 computer, in the Computing Center of the California Institute of Technology.

Besides furnishing further checks on the validity of the data obtained from the analog computer and the theoretical 2 impacts/cycle solution, it provided also (by the propagation of round off errors) convenient means of simulating the actual propagation of small perturbations in the steady state solution.

Among the basic features of this program were the following ones:

a) The RHS of Eq. (4. 5) was evaluated at $t = t_i + j \times \Delta t$ repeatedly (with j increasing by unity each time) until the quantity $\left(\frac{d}{2} - |y|\right)$

became negative. Then, depending on the rate of convergence of the iterations, the Newton-Raphson[28] or the bisection method was used to find t_{i+1} for which

$$\left| \frac{\frac{d}{2} - |y|}{d} \right| \leq \epsilon \tag{4.7}$$

where ϵ was usually chosen to be $O(10^{-6})$.

b) Notable among the troublesome cases encountered (in regard to satisfying Eq. (4.7)), were those involving consecutive impacts on the same side of the container, and when e was zero (i.e., no rebound after impact). Consequently, several tests were incorporated in the program in order to test, whenever Eq. (4.7) was satisfied, if that condition corresponded to a genuine impact, or if it was merely due to an undetected shortcoming of the program.

c) When a periodic solution would pass the test designed to determine if it had reached steady state conditions, the program would then discontinue that solution and start constructing a new one corresponding to a new set of the parameters ω, r, $\frac{F_o}{k}$, μ, δ, e, and $\frac{d}{F_o/k}$. Solutions that did not pass the steady state test were terminated after reaching a specified number of impacts.

d) Single precision arithmetic was employed throughout the program which required, for $\frac{\Delta t}{\omega \rho} = 1$, an execution time of approximately 5 sec/100 impacts.

Table 4.1 shows a typical digital computer output, and how it compares to theoretical results predicted by the 2 impacts/cycle

Table 4.1 (a)

DIGITAL COMPUTER OUTPUT

$r = 1.25, \quad \delta = .10, \quad \mu = .40, \quad e = .20, \quad \frac{F_o}{k} = 1, \quad \omega = 1, \quad \frac{d}{F_o/k} = 3$

impact # (i)	t_i	x_i	y_i	$\dot{x}_{i+}$	$\dot{y}_{i+}$	$\frac{x_{max}}{A}$ $t_{i-1} \le t \le t_i$
1	4.88	-1.5000	1.5	-.8169	-.2578	- .9246
2	6.52	- .3079	-1.5	1.0210	.6468	-1.0500
3	8.93	.7072	1.5	-.3582	-.6165	+ .7856
4	12.27	.4476	-1.5	.5965	.4782	- .5437
5	14.33	- .3433	1.5	-.5449	-.4929	.4444
6	16.92	.2044	-1.5	.5949	.4969	- .5026
7	19.18	- .0778	1.5	-.6011	-.5152	.5511
8	21.74	.0674	-1.5	.5839	.5175	- .5437
9	24.29	- .1226	1.5	-.5691	-.5084	.5125
10	26.81	.1655	-1.5	.5703	.5015	- .4970
11	29.30	- .1617	1.5	-.5791	-.5024	.5045
12	31.80	.1355	-1.5	.5828	.5065	- .5166
13	34.32	- .1237	1.5	-.5809	-.5083	.5197
14	36.84	.1297	-1.5	.5778	.5073	- .5153
15	39.35	- .1394	1.5	-.5768	-.5058	.5111
16	41.87	.1422	-1.5	.5780	.5054	- .5108
17	44.38	- .1386	1.5	-.5791	-.5060	.5129
18	46.89	.1351	-1.5	.5792	.5065	- .5143
19	49.40	- .1349	1.5	-.5786	-.5065	.5140
20	51.92	.1366	-1.5	.5783	.5063	- .5131
21	54.43	- .1377	1.5	-.5783	-.5061	.5127
22	56.94	.1375	-1.5	.5786	.5061	- .5130
23	59.46	- .1368	1.5	-.5787	-.5062	.5133
24	61.97	.1365	-1.5	.5786	.5063	- .5134
25	64.48	- .1367	1.5	-.5785	-.5062	.5133
26	67.00	.1370	-1.5	.5785	.5062	- .5131
27	69.51	- .1370	1.5	-.5785	-.5062	.5131
28	72.02	.1369	-1.5	.5786	.5062	- .5132
29	74.54	- .1368	1.5	-.5786	-.5062	.5132
30	77.05	.1368	-1.5	.5785	.5062	- .5132
31	79.56	- .1369	1.5	-.5785	-.5062	.5132
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
92	232.87	.1369	-1.5	+.5785	+.5062	- .5132
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.

Table 4.1 (b)

THEORETICAL SOLUTION

$r = 1.25, \quad \delta = 0.10, \quad \mu = .40, \quad e = .20, \quad \frac{F_o}{k} = 1, \quad \omega = 1, \quad \frac{d}{F_o/k} = 3$

1) $\tau = \tau_1 = 2.4817$ radians $\qquad x_a = -0.1369 \qquad (\frac{x}{A})_{max} = 0.5132$

$$P = \begin{bmatrix} -0.257 & -0.220 & 1.329 & -0.530 \\ 0.135 & 0.491 & 0.617 & 0.247 \\ -0.176 & -0.641 & -0.501 & -0.323 \\ 0.721 & -0.214 & -1.148 & 0.486 \end{bmatrix}$$

Eigenvalues:

#	Real Part	Complex Part	Modulus
1	0.306	0.662	0.730
2	0.306	-0.662	0.730
3	-0.197	0.142	0.243
4	-0.197	-0.142	0.243

2) $\tau = \tau_2 = -1.5318$ radians $\qquad x_a = -1.5674 \qquad (\frac{x}{A})_{max} = 0.9654$

$$P = \begin{bmatrix} -0.257 & -0.220 & 1.329 & 0.459 \\ -18.724 & 6.082 & 30.680 & -11.352 \\ 24.423 & -7.933 & -39.713 & 14.806 \\ -14.585 & 4.324 & 23.250 & -8.022 \end{bmatrix}$$

Eigenvalues:

#	Real Part	Complex Part	Modulus	
1	-42.919	0	42.919	> 1
2	0.505	0.609	0.791	
3	0.505	-0.609	0.791	
4	-0.0012	0	0.0012	

solution and its stability analysis.

4.5 Discussion of Results

The theoretical 2 impacts/cycle solution derived in Chapter 2 is compared in Fig. 4.16 with results obtained through the analog and digital computers.

In regard to Fig. 4.16, the following remarks can be made:

a) The two distinct curves shown (labeled τ_1 and τ_2) correspond to the two theoretical solutions obtained by using for the value of τ in one case

$$\tau = \tau_1 \equiv \tan^{-1}\left[\frac{-2\rho + H\sqrt{H^2+4-\rho^2}}{-\rho H - 2\sqrt{H^2+4-\rho^2}}\right] ,$$

and in the other

$$\tau = \tau_2 \equiv \tan^{-1}\left[\frac{-2\rho - H\sqrt{H^2+4-\rho^2}}{-\rho H + 2\sqrt{H^2+4-\rho^2}}\right] .$$

b) The ratio $\frac{x_{max}}{A} = (\frac{x}{A})_{max}$ was found by evaluating Eq. (2.26), expressed in the form

$$\frac{x}{A} = e^{-\delta\frac{\Omega t}{r}}\left[N(B_1)\sin\eta\frac{\Omega t}{r} + N(B_2)\cos\eta\frac{\Omega t}{r}\right] + \sin(\Omega t + \tau),$$

at $\Omega t = j \times \Delta t$, $(j = 0, 1, 2, \ldots)$, with $\Delta t \approx 0.01$, up to $\Omega t = \pi$.

c) The two solutions coalesce at the extremes d = 0 for which

$$\tau_1 = \tau_2 = \tan^{-1}(-\frac{H}{2}) ,$$

and at $\frac{d}{F_o/k} = 23.197$ (where $H^2 + 4 - \rho^2 = 0$), since then

$$\tau_1 = \tau_2 = \tan^{-1}(\frac{2}{H}) .$$

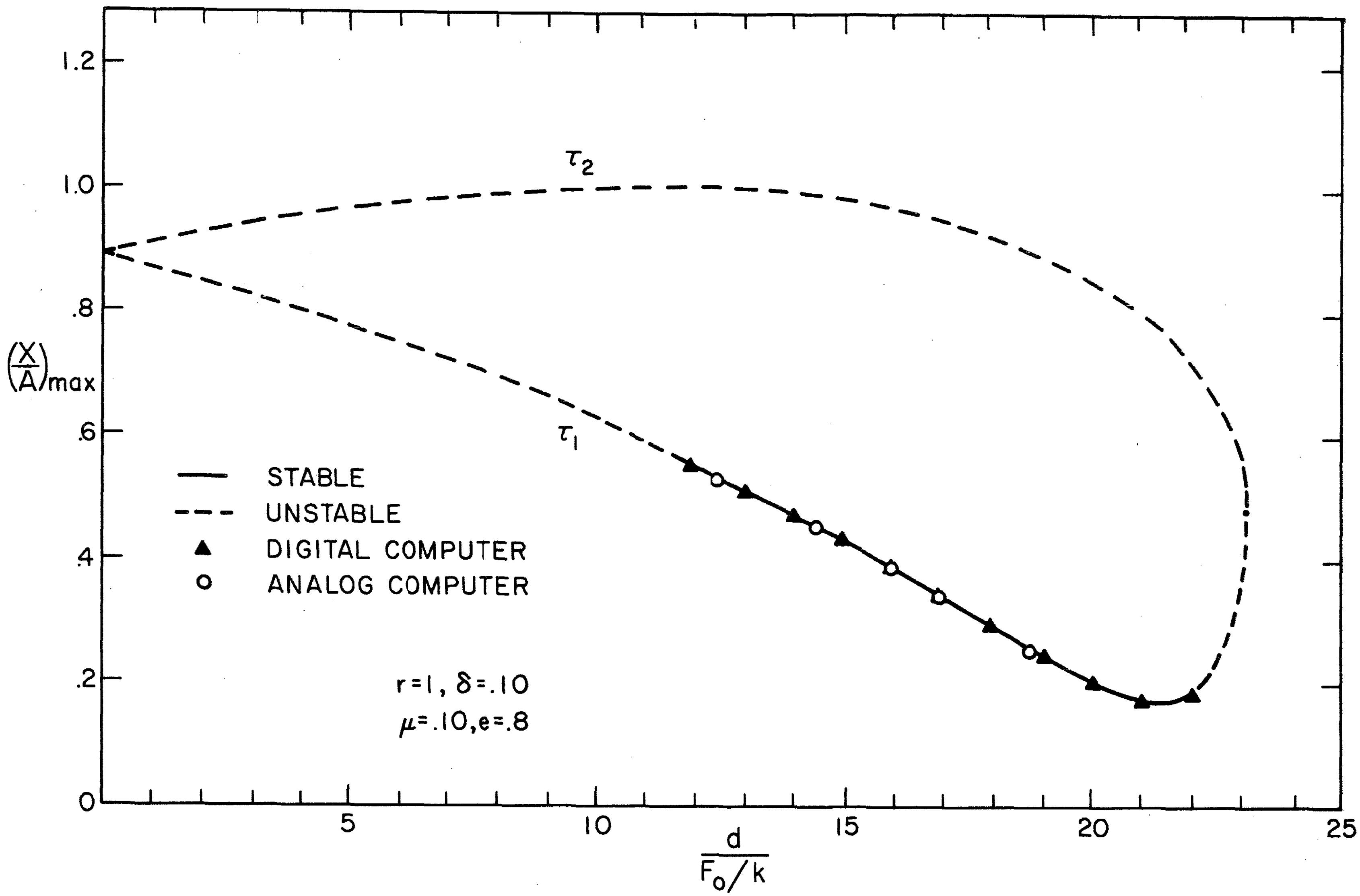


FIG. 4.16 SOLUTION CURVE

For $\frac{d}{F_o/k} > 23.197$, τ is complex; consequently our 2 impacts/cycle solution does not exist.

d) At d = 0, the same value for $(\frac{x}{A})_{max}$ will be found if the system is treated as a single degree-of-freedom oscillator with a natural frequency $\omega' = \frac{\omega}{\sqrt{1+\mu}}$.

e) The stability analysis indicates that the τ_2 curve is entirely unstable, while the τ_1 curve is only partly so.

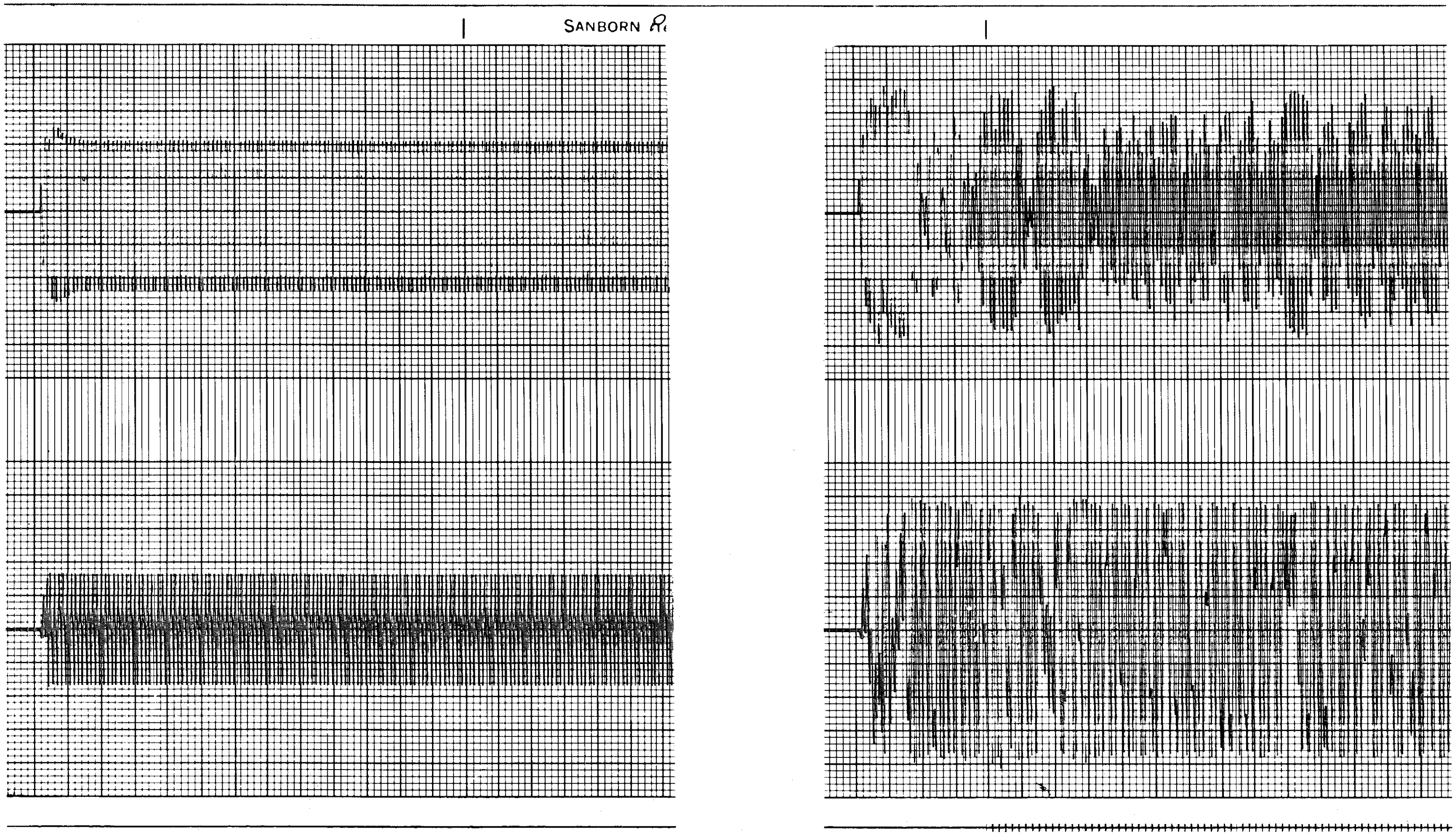
The stability boundaries were determined by the method described in Section 3.3.

f) Stable solutions obtained through the digital computer agreed with the theoretical solution and the stability analysis. Also, no symmetric (i.e., repeated at intervals of $\Omega t = \pi$) 2 impacts/cycle solutions were found outside the stable region.

g) Keeping in mind that the precision and accuracy of analog computer results are limited ($\approx 5\%$ in present case), the data obtained from such a computer is in satisfactory agreement with comparable theoretical and digital computer results.

h) Outside the stable region, both analog and digital computer results show that (see Fig. 4.17) for $\frac{d}{F_o/k}$ close to 11.56, the resulting motion is stable with 2 impacts/cycle but <u>not</u> symmetric. However, for $\frac{d}{F_o/k} > 22.25$ the motion is irregular.

The complicated functional dependence of the stability boundaries on the parameters of the system is illustrated, for typical cases, in Figs. 4.18 and 4.19, from which the following can be observed:



(a) $\frac{d}{F_o/k} < 11.56$

(b) $\frac{d}{F_o/k} > 22.25$

Fig. 4.17. Unstable (in the sense of symmetric 2 impacts/cycle motions (δ = .10, e = .80, μ = .10, r = 1).

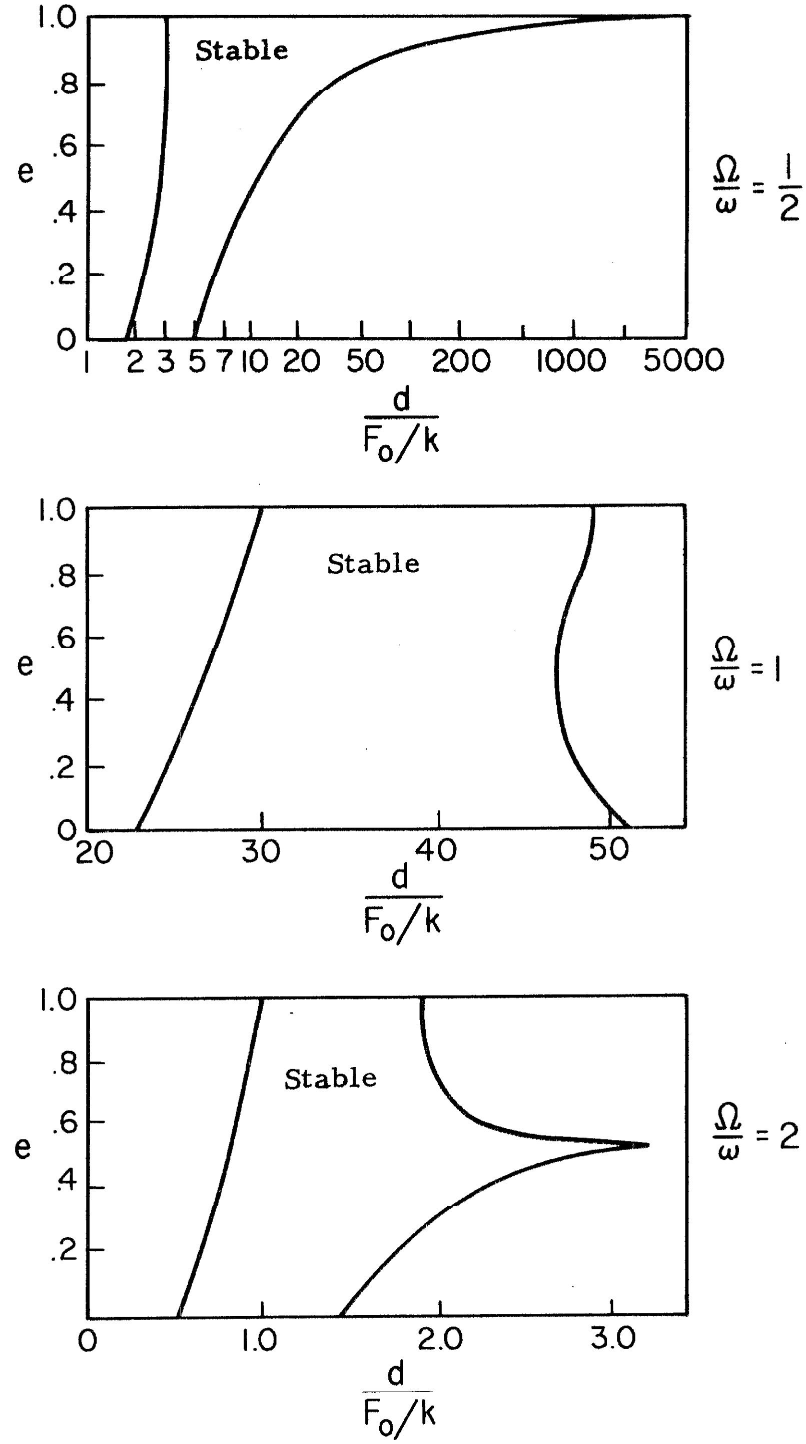


FIG. 4.18 STABILITY BOUNDARIES ($\delta = 0.01$, $\mu = 0.05$)

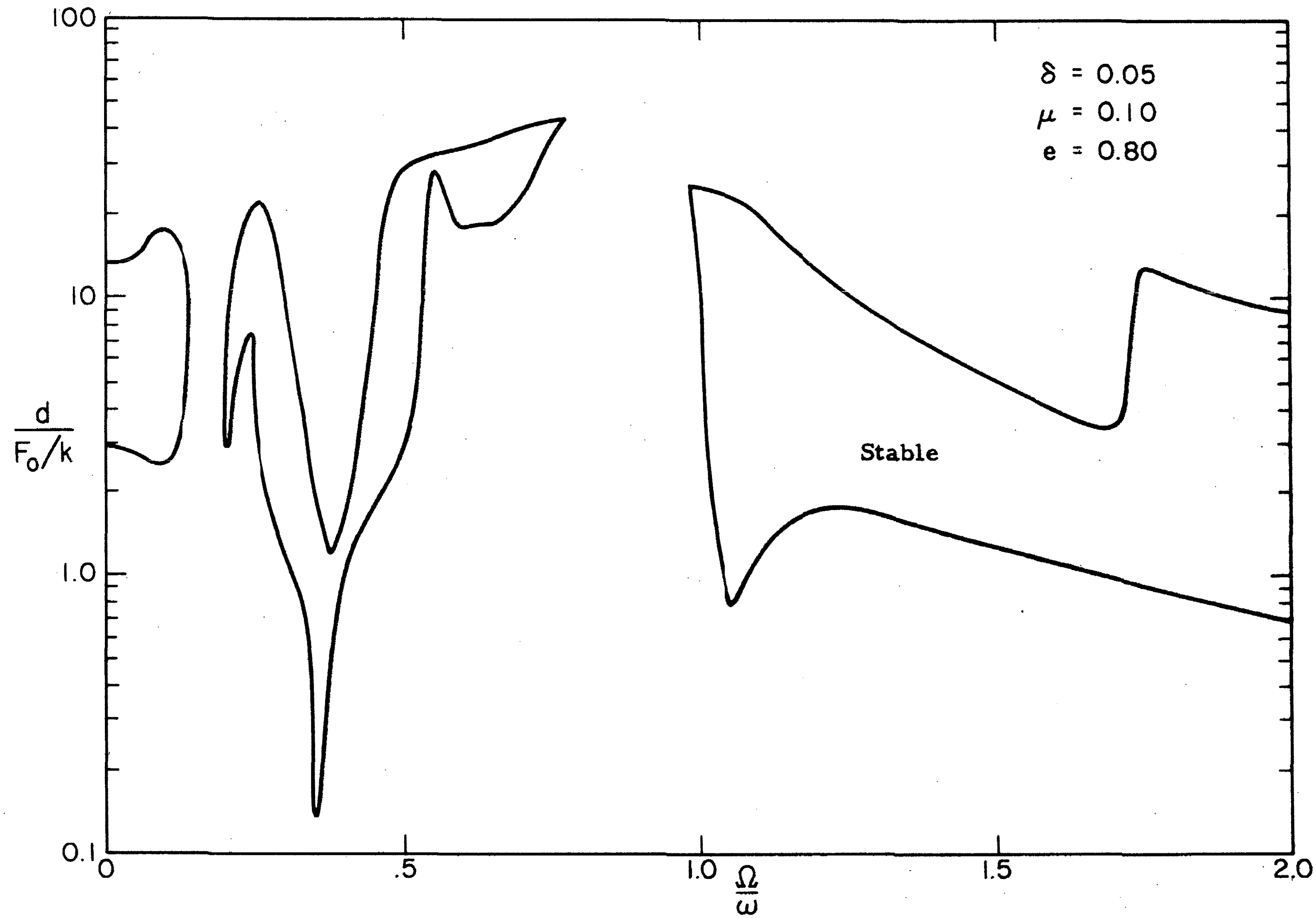


FIG. 4.19 STABILITY BOUNDARIES

a) The LHS of the three regions shown in Fig. 4.17 are qualitatively alike, and they correspond to the cases where one of the eigenvalues of P is real and its modulus is unity. The functional relation between the λ's and the parameters of the system is more complicated on the RHS boundary on which $\lambda_1 = \bar{\lambda}_2$, $|\lambda_1| = 1$.

b) The variation of e (usually the reduction of its value) with time can indeed cause stable 2 impacts/cycle motions to drift out of the stable regions.

c) The predicted behavior of the solution in various regions was found to be actually so. For example, with $r = 0.5$, $\delta = 0.01$, $\mu = 0.05$, $e = 1.0$, $\frac{d}{F_o/k} = 2600$, the theory predicted the existence of a stable solution with $(\frac{x}{A})_{max} = 16.13$, and this was verified through the digital computer.

d) Most of the stable solutions corresponded to $\tau = \tau_1$. Stable periodic motions with multiple impacts per cycle exist when symmetric 2 impacts/cycle motion is not stable. For example, with $r = 1$, $\delta = 0.10$, $\mu = 0.10$, $e = 0.80$, $\frac{d}{F_o/k} = 10$, stable unsymmetric 2 impacts/cycle motion occurs and its impact periods are $\Omega t = 2.131, 4.152$ (instead of each being π). However, with $r = 1$, $\delta = 0.10$, $\mu = 0.10$, $e = 0.25$, $\frac{d}{F_o/k} = 5$, the motion is periodic in $\Omega t = 3\pi$ with 4 impacts occurring at intervals $\Omega t = 2.821, 3.356, 2.233, 1.015$. Combinations of the parameters that did not give rise to stable periodic motions resulted in seemingly erratic behavior of the impact damper.

As a further test of the validity of the present analysis, let us consider the following cases.

Case 1: $\frac{\Omega}{\omega} = 1$ $\delta = 0.10$ $\mu = 0.401$ $e = 0.20$ $\frac{d}{F_o/k} = 5.065$

Case 2: $\frac{\Omega}{\omega} = 1.25$ $\delta = 0.10$ $\mu = 0.401$ $e = 0.20$ $\frac{d}{F_o/k} = 1.646$

For case (1), Grubin used his analysis to find that $\frac{x_{max}}{A} \approx 0.28$ and it agreed with his numerical solution after about 10 collisions. The present analysis predicts the existence of two solutions resulting in $\frac{x_{max}}{A} = 0.2749$ and 0.8063 corresponding to $\tau = \tau_1$ and τ_2, respectively, and that only the solution corresponding to $\tau = \tau_1$ is stable. This was verified by digital computer solution.

In regard to case (2), Grubin concluded that after 13 collisions 'the motion settled to steady values". [2] His conclusion does not agree with the present analysis which predicts that both 2 impacts/cycle solutions for this case are unstable. Step-by-step construction of the solution on the digital computer, beyond 13 impacts, resulted in an unstable 2 impacts/cycle solution. It thus appears that the stable result predicted by Grubin was a consequence of not carrying the solution far enough. This illustrates a basic difficulty of using such numerical solutions for stability investigations.

5. SUMMARY AND CONCLUSIONS

Periodic symmetric 2 impacts/cycle solutions were sought and their asymptotic stability boundaries were determined analytically. The stability analysis involved a perturbation of the phase space trajectory of the motion, and it indicated that the solution was stable if the modulus of all the eigenvalues of a certain matrix is less than unity. This matrix continuously related the perturbations immediately after each of two consecutive impacts. Results of the analysis were verified by:

a) Numerical step-by-step construction of solutions for all types of motion; however, due to round off errors in digital computations only stable solutions were found.

b) Experiments with an electronic analog computer.

c) Experiments with a mechanical model.

The following observations can be made on the basis of the present investigation:

1. For some parameters for which symmetric 2 impacts/cycle motion was not stable, stable periodic solutions with multiple impacts per cycle could be shown to exist.

Even for cases in which no stable periodic motions were established, the impact damper was often effective in reducing vibration amplitudes. On the other hand, for some stable periodic solutions, the impact damper resulted in an increase of vibration amplitude instead of a decrease. Stability alone is not the critical parameter for deciding the effectiveness of the impact damper.

2. As in the case of all nonlinear systems, the performance of the impact damper is dependent on the amplitude of the exciting force--unlike the linear Frahm dynamic vibration absorber.[29] In situations where the utilization of a dynamic vibration absorber is practical, and if the amplitude fluctuations of the excitation are not excessive, an impact damper can offer a tempting choice. While the requirements of a tuned linear system are exacting, the effectiveness of an impact damper is relatively insensitive to system parameters.

3. The theoretical solutions and stability analysis for periodic motions with a different number of impacts per cycle, or with a different period than the one treated in this thesis, may be obtained, with some effort, by extending the methods used here.

If the impact damper is looked upon as a highly nonlinear dynamic vibration absorber[30-33] and a proper damping mechanism is incorporated in the mathematical model to represent the effects of a finite coefficient of restitution, the method of slowly varying parameters may offer an alternative approach for treating the forced oscillations of the resulting two degrees of freedom system.[34]

4. Strictly speaking, if the mathematical model of Fig. 2.1 is started from a state of rest with the particle in the middle of its container, the impact damper will not operate if the ratio of the container clearance to the original amplitude of the primary system is > 2. In actual situations this condition will be remedied by the inevitable presence of friction between the particle and the primary mass, the initial displacement of the particle from the center of its container, or, when used, the effect of a weak coupling spring.

5. The impact damper appears to be most effective when used in conjunction with lightly or negligibly damped systems in resonant states. Under these conditions it can operate effectively with practical values for its parameters and usually it is not sensitive to slight changes in the parameters.

Since in practical applications the resulting amplitude rather than the existence of stable periodic motions is of prime concern, the impact damper fulfilled its role even when its motion was not steady. This was particularly true with undamped systems forced by a sinusoidal force at resonance, or subjected to random excitation. Essentially, the action of the damper disorganized the orderly process of amplitude buildup, thus reducing the response drastically.

6. The functional dependence of the stability boundaries, for any given set of the parameters, on the frequency ratio is complicated, especially for 0 < frequency ratio < 1. In the immediate vicinity of resonance (where the impact damper would normally be used) the stability boundaries enclose within them a sufficient range of system parameters to make stable symmetric 2 impacts/cycle motion practically realizable.

Needless to say, the present study does not exhaust the supply of interesting problems related to the behavior of the impact damper that await solution. For instance, one such problem is to prove that the response of the system to a bounded input is bounded. Also, in view of the effectiveness of the impact damper with random and impulse-like excitation, the feasibility of using impact damping to reduce, for example, earthquake induced or rocket vibrations merits investigation. Another practical area which should be further examined is that of the multiple-particle impact damper.

REFERENCES

1. Lieber, P. and Jensen, D. P., "An Acceleration Damper: Development, Design, and Some Applications," Trans. ASME vol. 67 (1945), pp. 523-530.

2. Grubin, C., "On the Theory of the Acceleration Damper," Journal of Applied Mechanics, vol. 23, Trans. ASME, vol. 78 (1956), pp. 373-378.

3. Arnold, R. N., "Response of an Impact Vibration Absorber to Forced Vibration," Ninth International Congress of Applied Mechanics (1956).

4. Warburton, G. B., Discussion of "On the Theory of the Acceleration Damper," Journal of Applied Mechanics, vol. 24, Trans. ASME, vol. 79 (1957), pp. 322-324.

5. McGoldrick, R. T., "Experiments with an Impact Vibration Damper," David Taylor Model Basin Report No. 816.

6. Lieber, P. and Tripp, F., "Experimental Results on the Acceleration Damper," Rensselaer Polytechnic Institute Aeronautical Laboratory, Report No. TR AE 5401 (1954). See also reference list of above report.

7. Sankey, G. O., "Some Experiments on a Particle or 'Shot' Damper," Memorandum, Westinghouse Research Labs (1954).

8. Duckwald, C. S., "Impact Damping for Turbine Buckets," General Engineering Laboratory, General Electric, Report No. R55GL108 (1955).

9. Estabrook, L. H. and Plunkett, R., "Design Parameters for Impact Dampers," General Engineering Laboratory, General Electric, Report No. R55GL250 (1955). See also reference list of above report.

10. Housner, G. W. and Hudson, D. E., Applied Mechanics - Dynamics, D. Van Nostrand Company, Inc., Princeton, N. J., Second Edition (1960), p. 91.

11. Timoshenko, S. and Young, D. H., Vibration Problems in Engineering, D. Van Nostrand Company, Inc., Princeton, N. J., Third Edition (1955), p. 82.

12. Bolotin, V. V., The Dynamic Stability of Elastic Systems, Holden-Day, Inc., San Francisco, Calif. (1964), p. 257.

13. Henrici, P., Error Propagation for Difference Methods, John Wiley and Sons, Inc., New York, N. Y., (1963).

14. Struble, R. A., Nonlinear Differential Equations, McGraw-Hill Book Company, Inc., New York, N. Y. (1962), p. 33.

15. Minorsky, N., Nonlinear Oscillations, D. Van Nostrand Company, Inc., Princeton, N. J. (1962), p. 5.

16. Phillips, E. G., A Course of Analysis, Cambridge (1962), p. 241.

17. Bellman, R., Introduction to Matrix Analysis, McGraw-Hill Book Company, Inc., New York, N. Y. (1960), p. 37.

18. Hildebrand, F. B., Methods of Applied Mathematics, Prentice-Hall, Inc., Englewood Cliffs, N. J. (1952), p. 61.

19. Wilf, H. S., Mathematics for the Physical Sciences, John Wiley and Sons, Inc., New York, N. Y. (1962), p. 20.

20. Beckwith, T. G. and Buck, N. L., Mechanical Measurements, Addison-Wesley Publishing Company, Inc., Reading, Mass. (1961), p. 95.

21. Harris, C. M. and Crede, C. E., Shock and Vibration Handbook, McGraw-Hill Book Company, Inc., New York, N. Y. (1961), vol. 1, p. 9-4.

22. Goldsmith, W., "Impact: The Collision of Solids," Applied Mechanics Reviews, vol. 16, No. 11, (1963), p. 855-866.

23. Tong, K. N., Theory of Mechanical Vibration, John Wiley and Sons, Inc., New York, N. Y. (1960), p. 23.

24. Jackson, A. S., Analog Computation, McGraw-Hill Book Company, Inc., New York, N. Y. (1960).

25. Jacobsen, L. S. and Ayre, R. S., Engineering Vibrations, McGraw-Hill Book Company, Inc., New York, N. Y. (1958), p. 174.

26. Korn, G. A., "A New Precision Dead-Space Limiter for Control System Instrumentation and Simulation," ACL Memo No. 30, Electrical Engineering Dept., University of Arizona (1961).

27. Johnson, C. L., Analog Computer Techniques, McGraw-Hill Book Company, Inc., New York, N. Y. (1956), p. 16.

28. Todd, J., Survey of Numerical Analysis, McGraw-Hill Book Company, Inc., New York, N. Y. (1962), p. 259.

29. Den Hartog, J. P., Mechanical Vibrations, McGraw-Hill Book Company, Inc., New York, N. Y., Fourth Edition (1956), p. 87.

30. Carter, W. J. and Liu, F. C., "Steady State Behavior of Non-linear Dynamic Vibration Absorber," Journal of Applied Mechanics, vol. 28, Trans. ASME, vol. 83 (1961), pp. 67-70.

31. Arnold, F. R., "Steady-State Behavior of Systems Provided with Nonlinear Dynamic Vibration Absorbers," Journal of Applied Mechanics, vol. 22, Trans. ASME, vol. 77 (1955), pp. 487-492.

32. Pipes, L. A., "Analysis of a Nonlinear Dynamic Vibration Absorber," Journal of Applied Mechanics, vol. 20, Trans. ASME, vol. 75 (1953), pp. 515-518.

33. Roberson, R. E., "Synthesis of a Nonlinear Dynamic Vibration Absorber," Journal of the Franklin Institute, vol. 254 (1952), pp. 205-220.

34. Caughey, T. K., "The Existence and Stability of Periodic Motions in Forced Non-Linear Oscillations," Ph. D. Thesis, California Institute of Technology (1954), pp. 18-50.